Published in Sandra Lapointe, ed.,
Themes from Ontology, Mind, and Logic. Present and Past.
Essays in Honour of Peter Simons.
(Leiden: Brill Rodopoi 2015, pp. 17–58; www.brill.com/)
Grazer Philosophische Studien vol. 91.

Collections as One-and-Many – On the Nature of Numbers

Ingvar Johansson

Abstract Peter Simons has been rather alone in the modern philosophy of mathematics to argue that the natural numbers should be regarded as *properties of* multitudes or collections. This paper, however, sides with Simons, but it modifies his property view by adding the notion of imposed collection boundaries and accepting fictional collections. Although partly inspired by the Husserl of *Philosophy of Arithmetic*, Simons dismisses Husserl's talk of psychological acts of collective combination, but this paper saves them by dressing them in modern cognitive clothes. Hereby, a reasonable *partially* constructivist notion of the natural numbers emerges.

*****

> The most naive opinion is that according to which a number is something like a heap, a swarm in which the things are contained lock, stock and barrel. Next comes the conception of number as a property of a heap, aggregate, or whatever else one might call it.
>
> Frege, Review of Dr. E. Husserl's *Philosophy of Arithmetic* (Frege 1972 [1894], 323)

In the early 1980s, Peter Simons interrupted my innocent slumber about the notion of set and the possible views of numbers that I had been taught as a philosophy student in the mid-1960s. When reading his papers "Number and Manifolds" and "Plural Reference and Set Theory" (Simons 1982a, 1982b) something entered my mind that I have thought about now and then ever since. Also, I have with interest and sympathy followed how he has continued to work on these his early ideas. Now I have systematized my related reflections. I end up on his side against all the usual analyses of numbers, but nonetheless there is a gap between our views. Of course I find it unfortunate that his early-Husserl and early-Russell inspired view of numbers has not yet found a reasonable place in the discussions of mainstream philosophy of mathematics (Husserl 2003 [1891], Russell 2006 [1903]). Apart from the last section, I will discuss only the natural numbers, but mostly for simplicity's sake use the unqualified notion 'number'.

1. Peter Simons on Numbers

According to Euclid's once famous definition in *Elements Book VII*, a number is a multitude composed of units (Euclid 300 BC). Peter Simons' standing basic anti-Euclidean, anti-Platonist, and anti-Fregean idea about numbers is this:

> I maintain that […] number is a property of external things of a kind which I shall call *manifolds* [also: multitudes]. In this I shall basically agree with Husserl against Frege, but the theory involved will perforce take account of Frege's objections to such a theory. (Simons 1982a, 161)

Simons is both a naturalist (meaning: everything that exists exists in our spatiotemporal world) and a nominalist (meaning: there are only spatiotemporally localized particulars; there are no non-localized universals, propositions, sets, or numbers). I think the driving force behind his philosophy of numbers comes out well in this quotation:

> One of the biggest problems facing a naturalist is to account for mathematics, a vast, ancient and noble discipline with millions of practitioners and thousands of books full of results, many of them of great subtlety and beauty. (Simons 2011, 1)

A manifold in the quoted sense is something completely stripped of all relations between the objects that make it up; if there are pure relations as objects in the manifold, they are kept distinct from their relata. The very term 'manifold' used by Simons above is only one among several possible; some others are 'multitude', 'collection', 'class', 'aggregate', and 'plurality'. In his contribution to the Michael Dummett volume of the Library of Living Philosophers, "What Numbers Really Are" (Simons 2007), and later, Simons has instead of 'manifold' chosen to use the term 'multitude'. Nowadays he presents his old original idea by saying:

> *My view is that numbers are certain non-distributive formal properties of multitudes*. (Simons 2007, 233; hyphen added)

Being non-distributive means not being a property of each of the members of the multitude separately, but only of all of them collectively. Being a *formal* property does here not mean being a formal-logical property; it means being a property that can meaningfully be ascribed to radically different kinds of ontological entities. A multitude of pebbles (physical entities) can have the property of being seven, a multitude of toothaches (mental entities) can have this property, a multitude of propositions (abstract entities) can be seven, and so can a multitude of novel figures (fictional entities), but of these kinds of entity only a multitude of physical entities can have the non-formal properties of weight and volume.

Simons' multitudes are in several respects similar to sets and mereological sums, respectively, but it is the differences that now are of interest. Multitudes differ from both sets and mereological sums in being nothing over and above its members. If the members are concrete, so is the multitude. A set, on the other hand, is always an abstract object distinct from its members, and a mereological sum is always a mereological individual distinct from its parts (Simons 2011, 5–6). Furthermore, there is always a null set, and in some mereological systems a "null item," but there can never be an empty multitude. As Simons says: "I claim that you cannot accept the existence of each of the three individuals A, B and C but deny existence of the trio ABC. In this multitudes differ from sets and also from mereological sums" (Simons 2011, 15).

I think the differences can summarily be put like this: *a set is more than the sum of its members/elements, a mereological fusion is the sum of its members/parts, and a multitude is simply its members*.

The early Peter Simons says, as noted, that he "basically agree[s] with Husserl against Frege," but the phrase 'against Frege' is important. He does not in general agree with Husserl; to the contrary:

> [M]y chief disagreement with Husserl is over his contention that pluralities [multitudes] are *constituted* as such by acts of collective combination, and accordingly are higher-order, *categorical* objects. (Simons 1982a, 162)

From the start, Simons attempts to create an axiom system that can underpin his view of what numbers are. At first he calls it 'manifold theory' (Simons 1982b, 220), but later of course 'multitude theory' (Simons 2011, 14). However, there is here more than a change of name. For a long time he aligns with the logician S. Leśniewski, because he regards Leśniewski as (in contradistinction to Frege) accepting plural terms in his logic. At least at first sight, plural subject terms such as 'we', 'they', 'these Xs', 'the Xs', '*a*, *b*, and *c*', etc. refer to multitudes.

The early Simons says: "The existing system of logic which our system most nearly resembles is Leśniewski's Ontology, sometimes called the calculus of names" (1982b, 238). Today, however, Simons has said farewell to Leśniewski (Simons 2011); the story of how he met him is told in (Simons 2013b, 227–228). The crucial point is that Leśniewski's calculus allows only first-order multitudes, but today Simons thinks that some problems in the philosophy of mathematics cannot be solved without the introduction of multitudes of higher order. He argues that such multitudes allow his multitude theory "to provide models in a way similar to that of set theory" (Simons 2011, 2), and so ground a logical notion of *following from*. He claims, though, that his staunch nominalism remains the same:

> [U]nlike sets, multitudes, *of whatever order* [italics added], are concrete if their members are concrete, and are necessarily such that if their members exist, then they exist. So multitudes are nominalistically acceptable, and if all their urelements (ultimate individual members) are natural, so are they. This closes the gap in the naturalistic account of mathematics and allows a naturalist to be a formalist with a clear ontological conscience. (Simons 2011, 2–3)

Peter Simons and I have got along quite well, but never, as he once put it, "see[n] eye to eye on all topics metaphysical" (Simons 2013a, 517). Like him I am a naturalist, but unlike him I am not a nominalist. I am a realist, although not a Platonist; I think the universals I believe in exist in our spatiotemporal world and nowhere else. However, in contradistinction to the best known contemporary immanent realist, D.M. Armstrong (Armstrong 1978), I think there are *both* universals and tropes; tropes then being spatiotemporally localized instances of universals. This means that I am not as ontologically far away from Simons' special "nuclear" trope nominalism (Simons 1994) as an unqualified nominalist-realist labeling may indicate.

In the philosophy of numbers, I have because of our common naturalism always felt as being close to his views, and below comes my attempt to sort out where I agree and disagree with him. Let it be added that the views I will put forward are somewhat tentative, and that Simons, with his new acceptance of higher-order multitudes, seems at the moment to waver a little, too. I will not discuss his axiomatizations; only informally discuss the very concept of multitude (manifold, collection, class, aggregate, plurality) and its possible referents.

I completely agree with Simons' view that natural numbers should be regarded as being first-order *properties of* something. That is, not as being some kind of property bearing thing-like *objects* such as Platonic numbers, spatiotemporal numerals, extensions of concepts (Frege-unsaturated or not), and sets of sets. Furthermore, the properties in question are neither *structures* in the sense of modern mathematical structuralism (Bostock 2009, ch. 6.4; Horsten 2012, sect. 4), nor some sort of *relations* (Bigelow 1988, Michell 1993, Armstrong 2010, ch. 13), and nor *kinds* (Lowe 2006, 81–83). On the view to be defended, the natural numbers are in a curious sense monadic properties, although of course (number 1 aside) not of single things, but of multitudes. Let's call it *the property view of the natural numbers*.[1]

This view does of course allow talk of numbers in abstraction from the relevant property bearers, the multitudes. It is no more odd to say that 7 lies in-between the numbers 6 and 8, without mentioning a property bearing multitude, then it is to say that typical red lies in-between the colors light red and dark red without mentioning a property bearer such as a color spot. Furthermore, as soon as operations on numbers are allowed, then the numbers may themselves in turn be ascribed (operation-dependent) properties, as in the sentence '5 is a prime number'. The property view of numbers does not cancel the distinction between pure and applied arithmetic, a fact that will come out more clearly later (Section 6).

With respect to how I differ from Simons, I have already said that I have a realist and Simons a nominalist view of properties, but I also differ from him in being sure that one can make semantic-philosophical sense of talk of fictional objects.[2] More precisely, I think that one can allow also multitudes of fictional objects as property bearers for numbers (Section 7). However, the most important difference between us in relation to the philosophy of number is that whereas Simons completely dismisses Husserl's collective combinations, I will save them by dressing them in modern cognitive clothes.

2. The Naturalist Husserl on Numbers

In what follows, partly out of respect for Peter Simons, I will not use his preferred term, 'multitude', and thereby try to add new connotations to it. Instead I will talk about *collections*. My views come much closer than Simons' to those put forward by Edmund Husserl in *Philosophy of Arithmetic* (Husserl 2003 [1891]); a book that in most of the twentieth century has suffered from a serious misinterpretation (see quotation below). I guess it is because of this fact that the book is not even mentioned in ordinary introductions to the philosophy of mathematics; see e.g. (Körner 1960, Bostock 2009, Horsten 2012).[3] In his recent book about Husserl's philosophical development, the Husserl expert J.N. Mohanty says:

> Ever since Frege's well-known review of the *Philosophie der Arithmetik*, it has been usual to ascribe to Husserl's work a wrong-headed psychologistic position, which – so the story goes – Husserl later on, partly under the influence of Frege's criticism, retracted. Recent researchers have shown that this story is mistaken. The thesis of *Philosophie der Arithmetik* is not "psychologistic" in the pejorative sense; it did not *reduce* numbers, for example, to subjective processes. […] Yet if the thesis is not psychologistic, it is not also what could be, in contrast, called purely logical. (Mohanty 2008, 4)

[1] This view is also argued for by (Yi 1998). He presents it by criticizing the relational account in (Bigelow 1988). He is aware of Simons as a forerunner, but in a long footnote (25) he complains about ambiguities in (Simons 1982a). I find him over-complaining. About Yi, see also footnote 12 below.
[2] My trope accepting realism is best defended in (Johansson 2014a) and my view of fictions in (Johansson 2010).
[3] Of course, noted by Simons: "The history of the philosophy of mathematics during the golden years of 1879–1939 hardly ever mentions Husserl" (Simons 2010, v).

Philosophie der Arithmetik was not translated into English until this century, and the translator comments:

> We can see, then, that Husserl made a disastrous choice of terminology in deciding to call the collective combination a "psychical" relation and describing it as having a "psychological nature." [---] Husserl was, in fact, *never* guilty of "Psychologism" with respect to numbers and their laws, nor in any sense in which he later rejected and refuted it. (Willard 2003, xxi and xxvii)

Readers who want an analytic-philosophically related introduction to Husserl's *Philosophy of Arithmetic* should read (Bell 1990, 31–84); Bell is of the same opinion as Mohanty and Willard.[4] The subtitle of Husserl's book is "psychological and logical investigations" (where 'logic' is understood in such a broad sense that it encompasses investigations of most kinds of abstract entities). In order to understand arithmetic epistemologically and ontologically, the early Husserl thinks that two combined kinds of investigations are needed. With this I agree. I will, though, exchange his views about psychological acts with views about cognitive activity in general. I will within block parentheses insert such a change directly in quotations, too.

3. Kripke's Lasso Catches and Husserl's Collective Combinations

Peter Simons takes the sets of set theory, whatever nature the elements have, to be *abstract* objects that do not coincide with their elements, whereas he regards multitudes to be nothing but all the elements, whatever nature they have. Here comes Cantor's famous characterization of sets (quoted from Simons):[5]

> By a 'set' we understand any collection into a whole *M* of definite and well distinguished objects *m* of our intuition or our thought, which are called the 'elements' of *M*.
>
> Unter eine "Menge" verstehen wir jede Zusammenfassung *M* von bestimmten wohlunterschiedenen Objekten *m* unserer Anschauung oder unseres Denken (welche die "Elemente" von *M* genannt werden) zu einem Ganzen. (Simons 2005, 143)

According to this characterization, a set is a constructed *whole* ("einem Ganzen"); it is the product of a collecting ("Zusammenfassung"). But after the collecting, the set is regarded as something in itself, moreover as an abstract object. In contemporary discussions about the nature of sets, the phrase 'Kripke's lasso' is sometimes used, e.g. by (Forster 2008). In such cases, it is also in contemporary set theory openly spoken of collecting processes. George Boolos says:

---

[4] Of the opposite view is a recent book on Husserl's early philosophy of mathematics, (Centrone 2010, xii). The author, however, says she "will approach Husserl's first book from a perspective that is orthogonal to the psychologism issue" (ibid.). Unfortunately, this also makes the book orthogonal to this paper.

[5] Already in *Philosophy of Arithmetic* Husserl calls Cantor a "mathematical genius" (Husserl 2003, 121n3). They were colleagues and close friends in Halle in the late 1880s and in the 1890s. Cantor was a member of the dissertation committee for Husserl's Habilitation thesis, which, later in revised form, became the first four chapters of *Philosophy of Arithmetic*.

> For when one is told that a set is a collection into a whole of definite elements of our thought, one thinks: Here are some things. Now we bind them up into a whole.* Now we have a set. We don't suppose that what we come up with after combining some elements into a whole could have been one of the very things we combined (not, at least, if we are combining two or more elements).
> * We put a "lasso" around them, in a figure of Kripke's. (Boolos 1971, 220)

When the lasso metaphor is accepted, the lasso must be said to fulfill two functions in relation to its catches. It functions as a boundary of the caught set, and it functions as something that turns the set into an abstract entity. I think it is good to have this contemporary constructivist or semi-constructivist view of sets in the back of one's mind when, now, I turn to the early Husserl's constructivist notion 'collective combination' ("kollektive Verbindung"), which, as said, Simons wholly dismisses.

Husserl takes great pains to show that there is a way of combining ("verbinden") objects into a unity, a totality (collection), which does not in any way rely on *any* kind of relations between the objects combined.[6] He summarizes: "When we think particular contents 'together' in the manner of a totality, this 'together' does not permit itself to be resolved into any other relations, so as to be defined by them" (Husserl 2003, 69). Sometimes, instead of 'combine' Husserl uses 'colligate' ("kolligieren"), a word that has almost left contemporary vocabulary. For Husserl, as Frege says in his review, "Collective connection [combination] is a relation *sui generis*" (Frege 1972, 322), and the relation is mind-dependent.

Next I will highlight Husserl's expression 'unitary representation of their totality':

> It was clear to begin with that the specific nature of the particular objects which are gathered in the form of a multiplicity ["Vielheit"] could contribute nothing to the content of the respective general concept. The only thing that could come into consideration in the formation of these concepts was the *combination* of the objects in the *unitary representation of their totality* [italics added; "einheitlichen Vorstellung ihres Inbegriffes"].[7] (Husserl 2003, 67)

Where the set theorists Cantor and Boolos talk about "collections into *wholes*," the philosopher Husserl talks about combinations into *unitary representations*, i.e., about unitary representations of collections. Here are two other typical quotations (and now starts my insertions of 'cognitive activity'):

> In such cases the contents are just simply thought [cognized] "together," i.e., as a totality. But in no wise are they truly disjoined or unrelated. To the contrary, they are *joined by means of the psychical act [cognitive activity] holding them together*. It is only that *within* the content of that act all perceptible unification is lacking [first italics added]. (Husserl 2003, 76)

> So testimony from many sources – and, above all, from inner experience [cognitive experience] itself – tells us that we must decide in favor of the second view

[6] In (Centrone 2010), see note 4, Husserl's "kollektive Verbindung" is translated as 'collective *connection*' instead of 'collective combination'. I don't think this is good, since 'connection' easily gives rise to the false view that there are pre-given or created relations *between* the objects of a collection.

[7] The relation between 'totality' and the German 'Inbegriff' is at another place in a footnote commented on by the translator: "The point, of course, does not come through in English. 'Inbegriff,' which we translate as 'totality,' involves *Begriff* or 'concept,' which in turn involves *greifen* or 'to grasp'"; Willard in (Husserl 2003, 100n2). That is, the constructivist element that Husserl builds into his very notion 'Inbegriff' is lost in the translation into 'totality'.

> mentioned, according to which collective unification ["Einigung"] is not intuitively given *in* the representation content, but instead has its subsistence only in certain *psychical acts [cognitive activity] that embrace the contents in a unifying manner* [the second italics added; "welche die Inhalte einigend umschliessen"]. (Husserl 2003, 77)

The terms 'psychical act' and 'inner experience' as here used are essential to the charge that in *Philosophy of Arithmetic* Husserl is defending a reductively psychologistic view of numbers. However, what it amounts to is only psychologism (cognitivism) with respect to the unification of non-connected objects or contents. After the unification, it is by no means for the unifier to decide how many objects the totality (the collection) contains.

The activity of collective combining is only one of two kinds of activities that Husserl finds necessary for the apprehension and constitution of numbers. The other kind of psychic (cognitive) activity is a certain kind of abstraction, one in which abstraction should not be understood as any literal taking away of contents. He specifies: "To disregard or abstract from something is merely to give it no special notice" (Husserl 2003, 83). For instance, if we have cognitively unified a chair, a flower, and a stone into an apprehended collection of three concrete things, we can then stop to give "special notice" to the features that ground their different classificatory concept labels, and simply regard them as three "somethings" (the scare quotes are Husserl's).[8] Husserl is using "something" ("etwas") in a sense that corresponds well to the contemporary philosophical term 'entity', but I will in this section continue to use his "something." Since this term can have anything whatsoever as its referent, it lacks ordinary differentiating conceptual content; and this is one of the reasons behind Husserl's rejection of Frege's view that numerical predications about things in the world are assertions about concepts.

Husserl's view of the natural numbers that he regards as perceivable (i.e. those not larger than twelve (Husserl 2003, 202)) looks like this:

> Let us look once more, then, at the psychological [cognitive] foundation of the number concepts.
>
> *According to our view two things make up the concept of number: 1) the concept of "collective unification [combination]" and 2) the concept of "something."* (Husserl 2003, 355)

Husserl does not regard the mentioned psychological investigations as being purely scientific-psychological. They are at the same time also both epistemological and ontological; let me explain the epistemological aspect first.

At the time, Husserl is an empiricist, although of an almost forgotten (Brentanian) breed. He thinks like Humean empiricists that all meaningful concepts must have some foundation or grounding in the perceptually given, but unlike them he is not a nominalist. He thinks that there are both universals (called 'abstract concepts' and 'abstracta') and instances of universals (sometimes called 'Moments' and sometimes 'conceptual objects').[9] I think an analogy

[8] Frege makes in his review the following remark about Husserl's contentless "something": "From each object we finally derive something which is completely without restrictions on its content; but the something derived from the one object nevertheless does differ from that derived from the other object, although it is not easy to say how" (Frege 1972, 324). It seems to me as if Frege has not really taken seriously how Husserl wants his abstraction procedure to be understood; in fact there are "restrictions on content" that make the objects differ, but they are not attended to. To Husserl, this notion of "something" seems to have been the point of departure for a lifelong grappling with what it means to be a formal concept; about this see (Mulligan 2013).

[9] In *Logical Investigations* (Husserl 1970) Husserl uses other names for universals: 'Species', 'Idea', and 'universal object' (Johansson 2010).

between Hume and Husserl can make clear what Husserl's empiricism in relation to numbers amounts to.

According to Hume, all truly meaningful ideas are at bottom founded on simple impressions. This fact, however, does not for him rule out the possibility of meaningful representations of non-existing entities, i.e., of fictions such as "winged horses, fiery dragons, and monstrous giants" (*Treatise* 1.1.3.4). His view requires only that such complex ideas can be resolved into simple ideas that once have had simple impressions as their referents. Moreover, he even allows as meaningful certain simple ideas that have not been preceded by any corresponding simple impression, if only their possible referent is meant to be closely similar to some earlier simple impressions (among Hume scholars, the famous case of "the missing shade of blue"). Hume posits a faculty of imagination that he ascribes the following abilities: to create ideas that are copies of impressions, to create ideas of the missing-shade-of-blue kind, to unite impressions and ideas into larger complexes, and to separate such complexes into their parts.[10]

In analogy with Hume's distinction between ideas preceded and not preceded by directly corresponding impressions, Husserl distinguishes between two kinds of number concepts, *authentic* and *symbolic* ones; the latter have to be grounded in the former in order to be meaningful representations. Authentic number concepts (the natural numbers ≤ 12) can as referents have phenomena that can be found in perceptual experiences when the two operations of collective combination and abstraction into "somethings" have modified a given perception. Symbolic number concepts, on the other hand, are created by arithmetic operations on the already given authentic number concepts, and can *only* be "representations by means of signs" (Husserl 2003, 205). Based on this distinction, Husserl divides *Philosophy of Arithmetic* into two parts; the first is called "The authentic concepts of multiplicity, unit and whole number," and the second "The symbolic number concepts and the logical source of cardinal arithmetic," respectively. Just like Hume, Husserl takes a faculty of imagination for granted.

What then is Husserl's ontological stance with respect to the referents of the authentic number concepts? As far as I can see, the most reasonable interpretation is this. Both the mind-independent world and the primordial perceptual field contain multiplicities of various kinds, but they do not in themselves contain any numbers. The existence of such multiplicities, however, is a necessary condition for the existence of numbers, but there are two other necessary conditions, too. There must also be results of the operations of collective combination and abstraction (the "somethings"). Jointly, these three necessary conditions (multiplicity, collective combination, "somethings") make up a sufficient condition for an authentic natural number concept to be applicable. Husserl's view implies that the referents of authentic number concepts contain one objective part, that of a given multiplicity, and two created subjective-psychological parts (purely cognitive parts).

Equally important to Husserl's analysis of numbers is a distinction between *abstract* and *general* concepts, which he accuses Frege for overlooking (Husserl 2003, 175). I understand him as identifying abstract concepts with universals and general concepts with general terms nominalistically understood, i.e. as mere names for multiplicities of particulars; the general concepts are also called 'general *names*'. He never himself pauses to explain the distinction, but he is very explicitly claiming that the referents of number concepts are not abstract concepts (universals). In a section called "The numbers in arithmetic are not abstracta" he states:

> Thus, 5 does not signify the concept (the abstractum) *five*; but rather 5 is a general name (or else a calculational sign) for any arbitrary group as one falling under the [general] concept *five*. (Husserl 2003, 191).

[10] The details of my interpretation of Hume can be found in (Johansson 2012).

And in the preceding chapter he has said:

> Considered formally, number and concrete group are related as are [general] concept and conceptual object [instance of universal]. *Thus number relates, not to the [abstract] concept (Begriff) of the enumerated objects, but rather to their totality (Inbegriff).* (Husserl 2003, 174)

That is, the authentic (general) number concept 5 does neither in pure nor in applied arithmetic refer to a universal, but always to a totality of "somethings." One of his reasons is that if a number concept had a number universal as its referent, then, since there is by definition only one universal of each kind, an addition such as 5 + 5 would make no sense (ibid., 174–175). This observation functions as a *reductio ad absurdum* of all kinds of number Platonism.[11]

If one already has recourse to the natural number concepts, then one can of course say that a given concrete multiplicity must have as many members as it, after the abstraction procedure, has "somethings." If we have 5 pebbles, we have also 5 "somethings"; and vice versa, if we have 5 "somethings" that happen to be pebbles, we also have 5 pebbles. Husserl's point is that the number concept *directly* connects only to the collective combination of "somethings," and only indirectly via the "somethings" to the collective combination of the pebbles. About this fact, he is very outspoken (Husserl 2003, 174). His view can in the linguistic mode of presentation be put like this: *The sentence 'There are 5 pebbles' is shorthand for 'There is a collective combination of 5 "somethings," and all these "somethings" are pebbles'.*

Ontologically, authentic numbers, i.e., referents of an authentic number concept, can be found in perceptions modified by the operations of collective combination and abstraction. Husserl's view of *symbolic* number concepts can now, taking the Hume analogy for granted, easily be stated. He finds, like Hume, no general semantic problem with concepts that may have merely fictional referents, if only in some way or other the words and symbols used can be linked back to some perceptual phenomena. Therefore, ontologically, symbolic number concepts can refer both to collective combinations of spatiotemporally existing "somethings" and to collective combinations of fictional "somethings."

Simons claims, I repeat, that natural numbers are *formal properties* of multitudes. Husserl comes in my explication close to saying that, first, natural numbers are *properties* of collective combinations; and since the relevant such combinations are combinations of "somethings," he also, second, comes close to saying that numbers are *formal* features of collective combinations of concrete entities. However, he never says either of these things, not even implicitly. Therefore, there is a notable difference between Simons and Husserl. Simons posits *concrete multitudes that lack a collective combination,* and he claims that they have a number *as a property*; Husserl posits *concrete multitudes partly constituted by a collective combination,* which as such he ascribes numbers, not saying whether the numbers should be regarded as properties or as something else. I think there is a position possible in-between Simons' and Husserl's; and a philosophically more reasonable one for that.

I think that Husserl in his way, and Simons in his, has overlooked two connected facts. The first fact is that Husserl's collective combinations and Simons' multitudes (what I call 'collections') must on closer analysis be regarded as simultaneously being both one and many, or being one-and-many. Second, in order to see this, one has to note that collective

[11] An analogous argument, it might be noted, has been given for the view that so-called *structural universals* cannot be analyzed as structures *of* universals, but requires the introduction of tropes (Campbell 1990, ch. 2.7). To be a structural universal in the sense at issue is to be a property that can be characterized by the form 'having n constituents of kind K' ($n > 1$). To be a water molecule, for instance, is to have the property of having 2 hydrogen atoms as constitutive parts, i.e., we have 1 + 1 hydrogen atoms. I find both Husserl's and Campbell's arguments valid.

combinations, multitudes, and collections need an imposed boundary in order to come into existence. Husserl explicitly regards collective combinations as partly psychologically [cognitively] constituted, and Simons implicitly regards multitudes as self-constituting ("they require nothing more for their existence than the existence of their members" (Simons 2011, 17)), but there is a more credible third position. It will be presented in the next two sections.

4. Collections as One-and-Many

The heading above alludes to an old subsection heading of Peter Simons: "Classes as many and as one" (Simons 1982b, 200). The latter distinction is introduced by Russell in *Principles of Mathematics*:

> In the preceding chapter we regarded the class [collection] as formed by all the terms [members], but usage seems to show no reason why the class [collection] should not equally be regarded as the whole composed of all the terms [members] in those cases where there is such a whole. The first is the class [collection] as many, the second the class [collection] as one. (Russell 2006 [1903], 139)

The "usage" Russell is referring to, he has exemplified with words such as 'the army' (compare 'the soldiers'), 'the navy' (compare 'the sailors'), and the Cabinet (compare 'the Cabinet Ministers') (Russell 2006, 68). Read in isolation, the quotation above may invite the view that classes [collections] are entities that, necessarily, have two aspects, a unitary aspect (the class as one) and a plural aspect (the class as many). This comes close to the kind of view of collections that I soon shall try to make sense of, but it is neither Russell's nor Simons'. To both, a class as many and a class as one are simply two distinct objects, not two necessary conditions for a collection [class, multitude] to be able to exist. I endorse the following view of Husserl: "It is a fact that we are often in a position to conceive of one and the same object as one and as many, as we wish" (Husserl 2003, 163).

According to Simons (leaving from now on Russell aside), classes as one are the abstract objects that set theory takes care of, and classes as many are the multitudes he wants to highlight and create an axiomatization of. I find Simons' view of the distinction unsatisfactory, and I will use the rest of this section to explain in what way I find it so.

Trivially, where there are several collections, the number of them can in principle be counted. And since they can be counted, there must be some sense in which they are not only collections as many, but also collections as one. It is only *as one* that each collection is counted. Therefore, I shall first try to make clear what I want the term 'collection as one' to designate, and then judge whether collections as one can be neglected the way Simons neglects them.

Many collections allow for a distinction between qualitative and numerical identity. When a collection is defined partly or wholly by an intension (i.e., defined as a collection of Xs, where X is a concept or description) this is always the case. For example, if someone talks about a certain collection of five pebbles, then the mere way of speaking indicates that there can be other individual collections falling under the concept 'five pebbles'. If two collections of five pebbles are picked out, then there are two numerically different but qualitatively identical collections; or, in other words, two collections of the same kind.

On the other hand, when a collection is defined by an enumeration, there is seldom need to talk about anything else than the numerical identity of it. Think, for example, of a collection that a speaker picks out by the expression 'this chair, this flower, and this stone'. Even if the collection conforms to the description 'a chair, a flower, and a stone', the speaker may have meant the collection of the three entities pointed at independently of whether they have been correctly classified or not.

In what follows, if nothing to the contrary is said, when talking about the identity of a collection, I am talking about its numerical identity, even though of course its qualitative identity may be of relevance for the numerical identity. Normally, in talk of collections, it is taken for granted that the members have their identity independently of the collection to which they belong, and this presupposition I will stick to, too. In this and the next section, I will discuss only collections whose members are spatiotemporal particulars, i.e., ordinary things, events, processes, states of affairs, tropes, and whatever.

From where does the identity of a first-order collection of spatiotemporal particulars stem? If one thinks of a collection such as the collection of *all* water molecules, it may seem simple. The identity stems from the nature of the members of the collection. But such an answer cannot possibly be the right one when it comes to a collection of, say, 100 000 water molecules. There are many such collections, and their members are both of the same nature and are equally many. Therefore, the numerical identity of the collections must be given in some other way than by the nature and number of their members. The same conclusion is even more obvious in the case of collections defined by enumeration. For instance, the members of the mentioned collection of a chair, a flower, and a stone have no nature in common. With the identity question now posed – and kept in mind in the background – a brief detour to language and logic will, I think, be illuminating.

In this century's philosophy, the term 'plural' has come to figure prominently in quite a new way. Here are three book titles (the asterisks are added): *Logic & Natural Language: On *Plural Reference* and Its Semantic and Logical Significance* (Ben-Yami 2004), **Plural Predication** (McKay 2006), and **Plural Logic** (Oliver & Smiley 2013). And the last two authors have rightly noted and remarked: "Simons was a pioneer of plural logic" (ibid., 98). In Simons' mentioned 1982-papers, there is a connection between his views on natural numbers and his stress on terms with plural reference ('we', 'they', 'these Xs', 'the Xs', '*a*, *b*, and *c*', etc.);[12] and I will add some observations on this relationship.

Simons' view about numbers is that numbers are certain *non-distributive* formal properties of multitudes (collections). Therefore, let's look at a non-distributively predicative sentence such as 'They are five'; meaning there are five members in a certain collection. The sentence contains a plural subject term ('They') and a non-distributive predication of the referent ('being five'). Predications containing numerals, however, are by no means the only kind of non-distributive predications. Oliver and Smiley claim that there are at least nine different kinds of one-place collective predicates, of which the kind 'number' makes up only one; even if they place it in "pole position" (ibid., 115–7). I want, and need for my purposes, to point out that in an important respect numerical predications differ from many other kinds of non-distributive predication. I will use a quotation from McKay's book as my point of departure. He writes:

> For example, any of these [collective and non-distributive] predicates might be true of some people without being true of any one of them:
> They are shipmates (classmates, fraternity brothers)
> They are meeting together
> They lifted a piano
> They are surrounding a building
> They come from many different countries
> They weigh over 500 pounds

[12] The same stress on plural reference is also to be found in the paper (Yi 1998) that is mentioned in footnote 1.

> Such predications are a routine part of ordinary language use, yet standard systems of first-order logic provide no place for such non-distributive predication. (McKay 2006, 1)

Out of context, plural subject terms (e.g. 'they') always look as if they refer to a collection. In the numerical predication 'They are five', the predication does not change the collection-nature of the referent of 'they', but this is what happens in McKay's six chosen predications. When the persons referred to by 'they' are said to be shipmates or meeting together (the first two examples), they are no longer seen as merely non-related members of a collection; they become related to each other and become, at least linguistically, parts of a state of affairs.[13] Analogously, when the persons referred to by 'they' lift a piano or surround a house (the next two examples), they become regarded as functional parts of a temporary functional unit, and, consequently, are no longer a mere collection. Being regarded as coming from different countries (fifth example) is a bit more complicated, but it is also a case of becoming regarded as parts of a complex state of affairs. Weighing over 500 pounds together means that a property of each member is added together, which, once again, means that after the predication the referents of 'they' can no longer be seen as only a collection; they do now also appear as constituents of a state of affairs.

Put briefly, in McKay's sentences the plural subject term refers to a number of entities that are parts of either a state of affairs or a functional unit. But neither states of affairs nor functional units are collections; both kinds of entities are at least in appearance more than a collection. As said, out of context plural subject terms look as if they refer to a collection, but in context – and that is the only place where they have real referents – they need not necessarily do so.

It is part and parcel of non-distributive predications that the subject term in the sentence in question becomes treated as referring to a unity, and I am by no means denying this. I am stressing two other things. First, I am pointing out the fact that non-distributive logical predicates such as 'being shipmates', 'meeting together', 'lifting together', 'surrounding', 'coming from many countries', and 'together weighing over 500 pounds' present the persons referred to by the word 'they' as being parts of a unity that is presented as, so to speak, being a *new entity* constituted by the collection at hand. Second, I am pointing out that this is *not* the case in a non-distributive numerical predication such as 'They are five'. Here, the referents of 'they' are just as in McKay's examples treated as a unity, but nonetheless there is no new entity that is constituted by the collection. In numerical predications, the non-distributive predication unites (as in all such predications) the collection, but only in the sense that the collection becomes no longer treated only as many, but treated also *as one*.

My view can be put like this: In a sentence such as 'They are five', (i) the word 'they' highlights the collection *as many*, (ii) the non-distributive numerical predication 'are five' treats the collective referent of 'they' *as one*, and (iii) the sentence as a whole is about the collection as one-and-many.

Plural predications involving numerals are non-distributive, and they presuppose that a collection apart from existing as many exists also as one. This fact is the linguistic side of the observation I made early in this section: collections can be counted, and must therefore in some sense be not only collections as many, but also collections as one. When two collections are equinumerous, each *as one* has *as many* the same number of members as the other. Collections should be regarded as one-and-many.

[13] I write 'at least linguistically', since Simons has argued that, ontologically, there are no mind-independent states of affairs (Simons 2009). I am of the contrary view (Johansson 2004, ch. 3).

5. On the Nature of Collection Boundaries

I can now return to the question from where the identity of a finite first-order spatiotemporal collection stems. Since every finite spatiotemporal entity must in some sense have a boundary that demarcates it from its surrounding, the question is from where the boundaries of finite spatiotemporal collections stem; I am not, remember, concerned with the boundaries of their members.

There are several subtle philosophical questions involved in the notion of a boundary (Varzi 2008), but in order to discuss the boundaries of collections all of them need not to be dealt with.[14] From Varzi's article I pick the distinctions (soon to be explained) between (i) owned versus unowned boundaries and (ii) natural versus artificial boundaries.[15] Then I will add a distinction between (iii) manifest and virtual boundaries.[16] The important conclusion will be this: *all finite spatiotemporal collections have unowned and artificial boundaries that only need to be virtual*.

For unreflective common sense, probably, a boundary is primarily an entity between two entities; for instance, the black line that delimits the white surface within this square □ from its white surrounding. Such boundaries are by definition *unowned* boundaries. But, philosophically, they of course give rise to the question what the boundaries between the first boundary (the line) and what is inside and outside of it are like. In relation to solid material things, reflections on perceptions easily lead to the view that the boundary of a thing must be the utmost part of it; or, as Euclid and Aristotle said, a boundary of a thing is the "extremity" of the thing (Varzi 2008, sect. 1). This view might well, I think, be called the view of reflective common sense. According to it, a true boundary is always an *owned* boundary; the boundary belongs to and is a kind of part (perhaps infinitesimal) of the entity it delimits. However, in mathematics curious entities have seen the light. For instance, the boundaries of an open interval are necessarily external to the interval; they are *unowned* without there being anything between the interval and the boundary. The open interval of points between the numbers a and b on the number line cannot but be defined by letting the external points a and b be the boundaries, i.e., the interval is defined as the set of points $\{x| \ a < x < b\}$.

Look at the following seven spots: • • • • • • • . Among them, in the middle, there is a collection of *five* spots. The boundary of this collection cannot come from any of the five members alone, and to say that it comes from them collectively makes it a magic how these can make the exactly similar spots to the left and the right of the central five-spots collection to be outside of the collection. The boundary of the five-spots collection cannot be found within the collection, and must therefore be unowned. As far as I can see, the nature of the case makes this true of all finite collections. *All finite collections have unowned boundaries*. This is due to the fact that members of a collection have their individual identity independently of the collection to which they belong.

A *natural* boundary is a boundary grounded in some kind of pre-given discontinuity or gap between an entity and its surrounding. An *artificial* boundary is a correspondingly decision-grounded man-made discontinuity or gap. The perceptual system creates of itself an immense number of natural boundaries, which make it contain many discrete entities. A philosophically perplexing such boundary is that between two colored surfaces, e.g. the boundary between this black spot • and its white surrounding. It is perplexing in the sense that it is hard to become clear about whether the boundary is black (and owned by the spot), white (and owned by the

[14] Simons has written a paper about boundaries, too (Simons 1991), but it is not relevant for my discussion.
[15] The latter is often discussed also under the labels *bona fide* versus *fiat* boundaries (Smith & Varzi 2000).
[16] Normally, the contrary opposite of 'manifest' is 'latent', and the opposite of 'virtual' is either 'real' or 'true', but for various reasons I think the opposition 'manifest' vs. 'virtual' creates the most fruitful connotations in the context at hand.

surrounding), black-and-white (owned by both), or colorless (unowned by both). Surely, however, there must in some way or other be a boundary that makes the black spot distinct from the white surrounding. *Within* the homogenously colored black spot, on the other hand, all the possible parts need an artificial boundary in order to come into existence. Quite generally, parts of homogenous spatial wholes have to have their boundaries artificially created; but what to say about collections?

We can easily talk about collections such as 'the pebbles in the bowl', where the bowl functions as the unowned boundary of the collection. Since the bowl is man-made, it is an artificial boundary. The collection, however, would retain its identity even if the pebbles would be put in a natural pit. Since the boundary of the collection is unowned, it doesn't matter what kind of material thing we decide to take as its unowned boundary. As noted, the boundary can be changed, but the collection remains the same. Therefore, even the natural pit mentioned must be regarded as an artificial boundary of the collection. It is a man-made choice that makes the collection to be confined within the pit, and which does not allow it to have any pebble outside of the pit as a member. As far as I can see, the nature of the case makes this true of all finite collections. *All finite collections have artificial boundaries*.

Artificial boundaries can be either manifest or virtual. With a *manifest* boundary I mean a boundary that is either completely grounded in something mind-independent or grounded in discontinuities in perception. Man-made material walls, as well as lines on a map, can be manifest boundaries; also, both this perceived square □ and this perceived spot • have manifest boundaries. A good example of a *virtual* boundary is the boundary between two pieces of land before it has been marked off on either the ground or on maps, but nonetheless talked about and decided on.

In order to individuate a collection privately, it is enough to imagine for oneself the members as together having a boundary, and thereby constituting a collection, but in order to individuate a collection socially, one has to be able to talk about the delimitation in some way or other. It is only the latter kind of boundary that is of interest for this paper, since we are concerned with numbers within mathematics understood as an intersubjectively communicable discipline; and such boundaries need to be at least virtual boundaries. All private and virtual boundaries are artificial boundaries, but not vice versa.

When qualifying a noun, the term 'virtual' means being very nearly the kind of thing mentioned by the noun; implying that the differences are not especially important. For example, to have a virtual monopoly is to have a kind of monopoly, but not a true monopoly. In optics, ordinary mirror images are called virtual images. Such images can easily be seen, but they cannot be obtained on a screen; i.e., they can be seen but not touched, but it is the seeing that is important. In computer technology, virtual realities are simulated environments that to a person's consciousness seem to be real or almost real. What I call a virtual boundary is a boundary that can neither be touched nor seen, but nonetheless has an intersubjective reality.

A virtual boundary is a social fact. When saying so, I connect to John Searle's analyses of such facts (Searle 1995, 2010). He claims that the logical structure of institutional social facts that are grounded in material entities can be captured by this simple expression: *X counts as Y in C*.[17] In more detail it looks like this: *the material entity X counts as having the status function Y in the context or collective intentionality C*. Let me explain by means of an example.

Coin money fits the formula nicely. A ten euro coin is a piece of metal (X) that counts as, has the status of, and functions as ten euro (Y) in many European countries and in international business (C). In C, specific pieces of metal and paper are both described and perceived – with far-reaching consequences – as money. In a mind-independent world, the same pieces would have much more limited effects. From the point of view of nature, money is only a fiction or a

[17] I have in the review (Johansson 2011a) and in a number of other papers discussed Searle's philosophy.

simulation, but from the point of view of society, money is as real as anything can be. When a material thing counts as money in a community, it *is* money in the community; it is what it is regarded as.

In my opinion, the same kind of analysis can be applied to collections. If the pebbles p_1, p_2, p_3, p_4, and p_5 count as a collection in a certain community, then here they are a collection. If the virtual boundary that makes them a collection counts as a boundary, then it is a boundary. *Collections can be just as real as money; and, from the point of view of nature, just as unreal*. As coins have one foot in nature and another in the social realm, collections of material entities have one foot in nature and another in the social realm. The members of a collection are material entities with presumably material boundaries, but the boundary of the collection is artificial and has at least virtual existence.

This was my brief defense of the claim that *all finite collections have unowned and artificial boundaries that only need to be virtual*. Simons, however, treats his multitudes as if there is no problem with how they are individuated. From my perspective, thereby, he implicitly treats spatiotemporal multitudes/collections as if they have *owned and natural boundaries*. As I will show, this difference is of consequence. Let it only be added that I find it understandable that unowned virtual boundaries become neglected. Since they are unowned they do not belong to the entities they delimit, and when they are virtual they are from the point of view of nature fictions.

Simons says that “multitudes [collections] with the same members are identical” (Simons 2011, 5), and with this I do of course agree. But from this fact he draws the false conclusion that “Multitudes are purely extensional collections” (ibid.). On my analysis, they are not *purely* extensional; they have, and necessarily so, also at least a virtual boundary as a condition for their existence. They require, pace (Simons 2011, 17), more for their existence than the existence of their members.[18] Note, though, that no *specific* boundary can be an identity condition of a collection. Since the boundary is unowned, a collection can retain its identity even if the boundary is exchanged. I have already mentioned that one and the same collection of pebbles can have either a man-made bowl or a natural pit as its boundary. Now it is time to add that a collection can retain its identity even if a manifest boundary (e.g., the bowl or the pit) is exchanged for one that is only virtual.

Common sense does not bother about the ontological status of the boundaries of collections, and in everyday life people take it for granted that we can create collections of arbitrary kinds of entities. That is, it is taken for granted that the human psyche/brain contains some kind of faculty or mechanism of imagination whereby collections can be freely created. Of course, philosophers who accept the existence of such a faculty, e.g. Husserl, come to the same conclusion. Even Hume, who denied the existence of enduring egos, but nonetheless let a faculty of imagination play a large role in his philosophy, is of this opinion.[19] In *Treatise* (1.2.2.3) he distinguishes between true unities (simple impressions and simple ideas) and fictional unities. He writes:

> ’Tis evident, that existence in itself belongs only to unity, […] For by the same rule these twenty men *may be consider'd as an unite*. The whole globe of earth, nay the whole universe *may be consider'd as an unite*. That term of unity is merely a fictitious denomination, which the mind may apply to any quantity of objects it collects together. (Hume 2000, 25)

[18] I have argued that collections are not self-individuating entities, but I am of the same opinion in relation to states of affairs. An artificial boundary is needed in order to isolate one finite state of affairs from all the encompassing state of affairs. For more reflections on this view and its repercussion on views in cognitive linguistics, see (Smith 2000).

[19] In (Johansson 2012) I have shown the central role that the faculty of imagination plays in Hume’s philosophy.

Put in other words: any number and kinds of existing true unities that the mind collects together might be called a unity, but in many such cases the new unity must be considered fictitious. In those cases, put in my words, there are collections of real particulars, but the boundaries are artificial.

Epistemologically, my analysis of collections implies that a collection cannot be truly apprehended in any other way than as simultaneously being both a unity (a collection as one) and a plurality (a collection as many), i.e., as unity-and-plurality, as one-and-many. When the unity is in the foreground, the plurality is in the background, and vice versa. But both of them have to be there all the time.

Mostly, singleton sets are regarded as being distinct from their member, i.e. $\{x\} \neq x$. The view of collection boundaries that I have put forward affords a similar answer to the question whether one-membered collections should be regarded as being distinct from their single member. Simons says that he regards an individual "as a degenerate case of a manifold [multitude, collection]" (Simons 1982a, 186), and that he will "stretch the meaning" of the term 'multitude' to allow a single individual to be a multitude [collection] (Simons 2011, 6). On my analysis, there is a clear difference between the member of a one-membered collection and the collection: *the member and the collection have numerically different boundaries*. Moreover, these boundaries are also of different *kinds*.

Look at the perceptual unity of the black spot • . As a unity it has a natural boundary (be it owned or unowned), but as a one-membered collection it has another kind of boundary, namely an artificial *virtual* boundary. Even a collection with only one member has a boundary that differs from the boundary of its single member. That is, I agree with Simons that there are one-membered multitudes [collections], but contra Simons I think there is no need here to talk about "degenerate cases" or "stretched meanings," even if in everyday language no one talks about one-membered collections.

My general claim is that collections, because of their specific kind of boundary, can and should be described as being one-and-many. In the next section, the expression 'one-and-many' will be specified into expressions such as 'one-and-7' and 'one-and-5', which refer to collections with 7 and 5 members, respectively. In the same way, a single-membered collection can be referred to by the expression 'one-and-1'.

6. Higher-order Collections and Arithmetic Operations

Assume we have four collections, one with 2 members (one-and-2), one with 3 (one-and-3), one with 4 (one-and-4), and one with 5 (one-and-5). Connecting to a symbolism used in (Simons 2011), I will symbolize them [e,e], [e,e,e], [e,e,e,e], and [e,e,e,e,e], respectively. Here, the symbol 'e' is shorthand for the term 'entity', and the square brackets symbolize – for me, but not for Peter Simons – the boundary of the collection in question. Out of these four collections we can easily create two *second-order* collections; one consisting of the first two first-order collections, and another of the last two collections. Thereby we obtain: **[**[e,e] , [e,e,e]**]** and **[**[e,e,e,e] , [e,e,e,e,e]**]**. We can equally easily create a third-order collection of the two second-order collections: **[[**[e,e] , [e,e,e]**]** , **[**[e,e,e,e] , [e,e,e,e,e]**]]**. In other words, as there can be collections, there can be collections of higher order. As Husserl says:

> Consequently it is a fact that we have the capability of representing several totalities together as unified into *one* totality, without thereby their separate unifications being lost. We represent totalities whose elements are in turn totalities. In fact, even totalities of totalities of totalities are thinkable, etc. (Husserl 2003, 97)

From the views defended in the former section it follows, that independently of whether a collection is a first-order or an *n*-order collection, its constituting boundary is unowned and needs only to be virtual. The fact that the boundaries have such a character means that they can never turn the contained multitude/collection into a new kind of entity. That is, even higher-order collections of concrete entities are in a sense only concrete collections; something which ought to please Simons. He says, as earlier quoted: “multitudes, of whatever order, are concrete if their members are concrete.” From my own perspective, it even looks as if I have filled in a lacuna in Simons’ multitude theory, namely supplied an answer to the question how higher-order multitudes are constituted.

The really important point, however, is not the fact that we can form collections of collections of collections as we can form sets of sets of sets. The point to stress is that we can *simultaneously* cognize more than one collection level in a higher-order collection, and see that they are included in each other. Many-leveled apprehensions are well known and regarded as important in other areas, but as far as I know never stressed in the philosophy of mathematics. For instance, in fractal art, part of the seeing consists in seeing that the same geometrical shape comes back again and again with ever decreasing size; the shapes are contained in one another in something like the way the dolls of a Russian matryoshka doll are contained within one another. In conversation, verbal irony is necessarily a two-leveled apprehension. When saying the opposite of what one really means, one’s listeners must understand that there are two layers of meaning, an explicit surface meaning plus an implicitly present antithetical meaning that is the true message.

Such a view of many-leveled apprehensions of higher-order collections opens up new possibilities about how to look upon the distinction between addition and multiplication; first some words about addition and then about multiplication.

According to the traditional view, multiplication of integers can always be regarded as a more or less complex procedure of repeated addition; and, surely, $3 \times 4 = 3 + 3 + 3 + 3$. However, in the empirical science of number cognition, the view that multiplication must always be understood as repeated addition has been questioned. Here are two quotations:

> Their [the children’s] performance could not be explained by processes of repeated addition, [...] These findings provide evidence for an untrained, intuitive process of calculating multiplicative numerical relationships. (McCrink & Spelke 2010, 204)

> The hypothesis of conceptual discontinuity between additive and multiplicative reasoning receives further support from the fact that the children in the correspondence group made significantly more progress in multiplicative than in additive problems. (Park & Nunes 2001, 771)

Now, of course, it might be claimed that these empirical findings about how the brain works cannot falsify the view that, *from a logical point of view*, the operation of multiplication must be understood as being based on that of addition. With this claim I agree, but if it can be shown that also from a logical point of view multiplication *can* take place independently of addition, fundamental things have to be re-thought. And I think the existence of second-order collections makes a kind of multiplication without addition possible and logically understandable. I am by no means discussing what may correspond to multiplication processes in brains and computers.

On the view I am expounding, remember, there can be no natural numbers if there is no prior cognitive activity that creates boundaries that delineate collections as one-and-many. On this presupposition, the natural view of addition involves only first-order collections, and it looks as follows.

Take Kant's old example of an arithmetic incontestable truth: 7 + 5 = 12. Below is a list of four equality sentences related to that truth; the first belongs to pure arithmetic and the others to the highest abstract level of applied mathematics:

(a) 7 + 5 = 12
(b) 7 e + 5 e = 12 e, or: one-of-7 + one-of-5 = one-of-12
(c) [e,e,e,e,e,e,e] + [e,e,e,e,e] = [e,e,e,e,e,e,e,e,e,e,e,e]
(d) a collection of 7 entities + a (distinct) collection of 5 entities = a collection of 12 entities

Let me explain the list bottom up. In all cases the addition symbol '+' represents the operation of combining the two (explicitly or implicitly mentioned) collections on the left hand side to a new collection that is placed on the right hand side. From my perspective, addition is a sort of *boundary-reconstruction*.

I regard sentence (c) as merely a symbolization of sentence (d), and sentence (b) as merely a symbolization of sentence (c). A Platonist mathematician would regard (b) as an application of a truth about mathematical numbers, but according to the property view of numbers, (a) should be understood as being an abstraction from the truth stated by (b).

Only collections of things of the same kind, i.e., collections whose members are described by the same count noun, can be added. For example, (a collection of) 7 apples + (a collection of) 5 apples = (a collection of) 12 apples, but (a collection of) 7 apples + (a collection of) 5 bananas have no meaningful common sum; it is only a collection of seven apples and five bananas. However, the sum 7 fruits + 5 fruits = 12 fruits is as meaningful as that concerned with apples, and when using the most abstract count noun possible, 'entity', we can add collections of whatever kind of phenomena we want. 7 entities (whatever their concrete nature is) + 5 entities (whatever their concrete nature is) = 12 entities.

Of course, what I have said about the addition 7 + 5 = 12 (or: 7 e + 5 e = 12 e) can be repeated in relation to the addition 3 + 3 + 3 + 3 = 12 (or: 3 e + 3 e + 3 e + 3 e = 12 e). Out of the four first-order collections on the left-hand side, which have 3 members each, a single first-order collection of 12 members is created. It is now a matter of *repeated boundary-reconstructions*. In the square bracket collection symbolism introduced, it can be written:

[e,e,e] + [e,e,e] + [e,e,e] + [e,e,e] = [e,e,e,e,e,e,e,e,e,e,e,e].

Now comes my thesis, namely that the multiplication 3 × 4 = 12 need not necessarily be regarded as a truncated description of the addition procedure 3 + 3 + 3 + 3 = 12, which contains only first-order collections. *It can be, no doubt, but it need not be*. If we allow the possibility of two-leveled apprehensions of second-order collections, we can make an immediate one-stroke cancellation of all the boundaries of the first-order collections, but nonetheless still have a collection. The second-order collection simply collapses into a first-order collection. When we apprehend a second-order collection that includes the apprehensions of four first-order collections, we may immediately see:

[[e,e,e] , [e,e,e] , [e,e,e] , [e,e,e]**]** = [e,e,e,e,e,e,e,e,e,e,e,e].

That is, a second-order collection with 4 first-order collections, each of which has 3 members, can immediately be seen to have as many members as a first-order collection with 12 members. It seems to me as if this equality can be regarded as logically representing multiplication without addition. It is as possible here to go directly (by means of boundary cancelling) from the single *second-order* collection on the left-hand side to the single *first-order* collection on the right-hand side, as it is in the addition case to go (by means of repeated boundary reconstructions) from four first-order collections to a single first-order collection. When one apprehends the second-order collection, one apprehends *both* the second-order collection *and* the contained first-order collections. To apprehend a second-order collection is to have a two-leveled apprehension.

On this analysis, both multiplication as repeated adding and multiplication without adding contain boundary-reconstructions, but the reconstruction operations are of different kinds.

The positing of apprehensions of levels within second-order collections has also another interesting consequence. It does in a simple way explain the following fact, discussed in mathematical pedagogic (Yoshida 2009). In pure arithmetic, $3 \times 4 = 4 \times 3$, and there is no need to bother about what number is the multiplicand (first factor) and what is the multiplier (second factor). In applied arithmetic, however, things are different. If a child is told to calculate by multiplication how many apples there are in 4 baskets with 3 apples each, it should use the multiplication 3×4, three (apples) times four (baskets), but if there are 3 baskets with 4 apples each, it should use 4×3, four (apples) times three (baskets).

In the square bracket collection symbolism I have introduced, the difference mentioned is made visible as a difference between two different kinds of second-order collections. That is: the purely arithmetic expression '3×4' is abstracted from the applied arithmetic expression **[**[e,e,e] , [e,e,e] , [e,e,e] , [e,e,e]**]**, but the purely arithmetic '4×3' is abstracted from the applied arithmetic expression **[**[e,e,e,e] , [e,e,e,e] , [e,e,e,e]**]**. These second-order expressions are *as such* distinct, below symbolized as $_2{\neq}_2$ (the subscripts indicate that only second-order entities are related), which means that we can state that

[[e,e,e] , [e,e,e] , [e,e,e] , [e,e,e]**]** $_2{\neq}_2$ **[**[e,e,e,e] , [e,e,e,e] , [e,e,e,e]**]**,

even though these two distinct second-order collections are grounded in one and the same first-order collection, [e,e,e,e,e,e,e,e,e,e,e,e].

7. Spatiotemporal Collections and Fictional Collections

One problem for the view that natural numbers are properties of collections of concrete entities is how to make sure that one can talk about infinitely many natural numbers. Peter Simons' present solution is that this becomes guaranteed by the existence of higher-order multitudes/collections (Simons 2011, 16–17). After n-order collections there are (n+1)-order collections, and so on. However, since I already think a faculty of imagination is necessary in order to create first-order collections, not to speak of an infinite number of higher-order collections, I think the simplest way out is to posit *fictional collections* as bearers of the natural numbers of pure arithmetic. The members of such first-order collections can be regarded as being as eternal and unchangeable as Platonic entities are regarded by the Platonists. Some words about this possibility.

When numbers are talked or written about, it is normally taken for granted that different numerals such as the Arabic '3' and the Roman 'III' refer to the same number, and that number concepts belonging to different number systems that correspond to each other, such as 3 of the decimal system and 101 of the binary system, refer to the same number. It is taken for granted that whatever numbers are, they are not to be identified with the symbols and concepts used in order to refer to them.

The lone Arabic number concepts of the decimal system, 1, 2, 3, etc., cannot be used to refer to anything in the spatiotemporal world unless they are connected to a count noun or a word that functions as such. There are no pure numbers in the primordial perceptual field or in the mind-independent spatiotemporal world. This is shown by the fact that if one is asked to point at 5, one has to ask '5 of what?' If, then, the asking person says 'take flowers', one can point at a collection of 5 flowers; if the answer is 'take pebbles', one should point at a collection of 5 pebbles. However, whatever kind of concrete collection one points at, it will not be an eternal and unchangeable collection. This is to me the central reason why pure arithmetic with its seemingly eternal truths such as '7 is larger than 5' request either Platonist or fictional collection referents. Since I am a naturalist, I will only consider fictions.

Everyday life contains a clear distinction between real people of flesh and blood and purely fictional people presented in novels, movies, and TV-series. Only people of the first kind have a biological body that can become truly sick, truly suffer pain, and literally die. When people are held apart in this way, the fact-fiction distinction is unproblematic. However, in story-telling it can easily be made very fuzzy. In the literary genre of fictionalized autobiographies, so-called autofiction, it is impossible for all except the author to tell what is intended as describing real facts and as describing fictional facts about the main figures involved. If the reader cares, she has to make hypotheses about the autobiography the way natural scientists make hypotheses about nature. To propose a hypothesis is neither to assert that a certain presumed fact is real nor to assert that it is a fiction; putting forward a hypothesis is a specific kind of language act.

Our natural language allows us to distinguish between factual assertions, fictional assertions, and factual hypotheses. We are even capable of a fourth kind of "aboutness" language act, namely the mere entertaining of facts. That is, we can talk about something without treating it as being real, fictional, or the content of a hypothesis. The semantic-philosophical question to ask is not whether these distinctions are illusory or not; the semantic-philosophical question to ask is how they are possible. This question, however, I will not touch upon here. In my opinion, the philosophy of numbers can rest content with a mere acceptance of the fact that quite often we can identify, re-identify, and communicate about certain fictions. Therefore, it is possible to ascribe fictions a certain mode of existence, even though it is a mode that make them at bottom partly mind-dependent (Johansson 2010). This kind of existence-fictionalism must not be conflated with what today in the philosophy of mathematics goes under the name of fictionalism; the latter regards mathematical entities as straightforwardly non-existing and, therefore, mathematical statements as being false (Balaguer 2008).[20]

From a linguistic point of view, the dictionary meanings of words are the same independently of whether the words appear in factual, fictional, hypothetical, or mere-entertaining discourses. Similarly, sentences that represent language acts such as orders, promises, exclamations, and declarations have a lexical meaning that is independent of kinds of discourse. These facts explain why no re-reading is necessary if one has read a story as being factually true, but later is told that it is a complete fiction. Also, conversely, we can read something as if it is a fictional novel, but later be told that it is a true story. In both cases, we can immediately without any re-reading switch and see the story in the new light. Let me call this capability the *fact-fiction switch ability* (Johansson 2013). Turning factual assertions into fictional ones is always possible, but, of course, turning fictional assertions into non-absurd factual ones cannot always be done.

Likewise, as far as I can see, numerical expressions have the same lexical meaning independently of what kind of discourse they appear in. The sentence 'There are 5 pebbles' has the same lexical meaning in a fictional novel as in a factual report, and the purely arithmetic

[20] This is not an exhaustive classification of what falls under the term 'fictionalism'. For instance, in (Gullberg 2011) a "fictionalism with truth" (ibid., 151) is defended. It differs from my position in two main respects. First, she does not have a property view of numbers; the mathematical truths she argues exist are truths about structures. Second, she is relying on a modification of Carnap's distinction between questions that are external and internal to given conceptual frameworks, whereas I think it suffices to distinguish between spatiotemporal and virtual-fictional entities. The obvious question to ask Carnap and Gullberg is what ontological status they ascribe the seemingly absolute conceptual framework in which the external-internal distinction itself is made and presented.

sentence ‘7 > 5’ has the same lexical meaning when it is intended to be about fictions as when it is intended to be about Platonic objects. Using the fact-fiction switch ability, we can turn all old talk about Platonic mathematical objects into talk about fictions; the problem is only how to retain in the change the view that there are arithmetic truths. In other words, the problem is how to deny the second horn in this naturalist-mathematical dilemma: *If the referents of pure number concepts are eternal and unchangeable Platonic objects, we have to reject naturalism; but if they are fictional objects, we have to reject the view that arithmetic contains truths*.

Of course, if we allow fictional stories to contain logical contradictions, then it is impossible to defend the view that the statement ‘7 > 5’ is true, since the contradictory opposite ‘¬(7 > 5)’ can be true, too. Therefore, the view to be discussed is whether non-contradictory arithmetic can contain truths if it has fictional referents, i.e., if the natural numbers are regarded as fictional entities, or, more precisely, as properties of fictional collections. My basic view is this. I think there are irreducible relations and operations, and if such there are, then assertions about relations and operations between fictional entities take on a special character.

If there are relational necessities stronger than that of physical necessity, then these make the corresponding relational assertions true independently of whether the relata are real or fictional. Look at these two triangles, ▼ ▾ , and call them L_1 and S_1, respectively. Obviously, L_1 is larger in perceived area size than S_1. It means that if two other perceived triangles exist that are exactly like these, then by some kind of strong necessity L_2 is larger than S_2. And the same must hold even if L and S are merely fictional triangles in a non-contradictory fictional story. If the relational sentence ‘L is larger than S’ is true by strong necessity, then it doesn’t matter whether the referents are spatiotemporal or fictional; or Platonic for that matter.

Likewise, if numbers are properties of collections, then relational sentences such as ‘7 > 5’ and ‘7 + 5 = 12’ are true independently of whether the collections that have the properties of being 7 and 5, respectively, are real or fictional. In spatiotemporally applied arithmetic, the natural numbers are properties of real collections of spatiotemporally finite and often changeable entities, but in fictionally applied arithmetic the natural numbers can be properties of collections of fictional eternal and unchangeable entities. This is my view.[21]

8. Natural Numbers and Real Numbers – Equinumerosity and Equiproportionality

Peter Simons has claimed that even if the discovery of Russell’s paradox made Frege’s analysis of the natural numbers impossible, one should note that if Frege’s logicism is extended to the real numbers, then other seemingly insuperable problems pop up (Simons 1987). Therefore, one may legitimately ask Simons – and those who, like me, tries to develop his property view of the natural numbers – how to look at the real numbers. I will conclude my paper with some remarks on this problem.

The property view of the natural numbers is tied to the notion of collections (or some synonym); and a collection is a collection of discrete entities. This means that the existence of the irrational numbers constitutes a problem for this approach just as much as it once did for

[21] Let me add one thing. Since I have the general view that properties exist within a determinable-determinate distinction (Johansson 2002), I regard each specific natural number as a determinate of a determinable that might be called ‘(natural) number’ or ‘multiplicity’. As red is a determinate of the determinable color, five is a determinate of the determinable number. If Husserl had made a distinction between determinable-determinate relations and genus-species relations, I think he would have put forward the same view. My reason is that the early Husserl can at least once be interpreted as claiming that the specific natural numbers are species of the genus cardinal number (Husserl 2003 [1891], 86), and that later in *Ideas* (Husserl 1982 [1913]) he becomes quite explicit about it: “cardinal number as such [“Anzahl überhaupt”] is a highest genus. Two, three, etc. are its *infimae species*.” Unfortunately, the English book here contains a misleading mistranslation. “Anzahl überhaupt” is translated into “any cardinal number whatever” (Husserl1982, 25).

the Pythagoreans. Of course, not even the rational numbers can when defined as fractions be monadic properties; they must be relations. However, since they can be regarded as relations between the number properties of two collections, they can at least in a simple way be grounded in monadic properties, but irrational numbers cannot. They require for such a grounding very complex mathematical operations; be it by Dedekind cuts, Cauchy sequences, or infinite series of nested fractions.

In order to illustrate the geometrical meaning of irrational numbers, I will use the two classic examples: $\sqrt{2}$ and π. The fact that the irrational numbers cannot in a simple ratio way be defined by rational numbers means that there is no extended entity (measurement unit), however small, that can make up a concatenated collection on both the hypotenuse (= $\sqrt{2}$) and the other sides (= 1) of an isosceles right-angled triangle; and there is no extended entity, however small, that can make up a concatenated collection on both the circumference of a circle (= πd) and its diameter (= d). If the measurement unit searched for covers one of the lines exactly, it cannot possibly cover the other; and vice versa. How, then, in what way should the property view of the natural numbers best be combined with a reasonable analysis of the real numbers? Simons says:

> The relationship between ratios of quantities of like kind and the real numbers with which they can be uniquely *correlated* [italics added] is many to one: the numbers are, *qua* objects of pure mathematics, different from all their many applications. Euclid in fact worked *solely* with ratios: it took until the 19th century with the work of Dedekind, Cantor and others for the pure mathematical theory of real numbers to be established. (Simons 2013, 532)

If such a correlation view were to be applied to the natural numbers, it would claim that the relationship between collections and the natural numbers is merely one of correlation, not that the numbers are properties of collections. This runs completely counter to the property view approach. Therefore, I think that this approach should not too quickly be combined with a correlation approach to the real numbers. If the natural numbers are properties of collections, then this fact ought in some way or other to spill over on what the real numbers are; or so I think. If the ontological nature of the natural numbers cannot be understood merely by means of collections and numbers-as-objects, then something like a wide property view ought to be true of the real numbers, too.

Simons' property view of the natural numbers says that such numbers are non-distributive formal properties of multitudes/collections. And I have defended it by adding to Simons' position the notion of imposed collection boundaries and by accepting fictional collections. My proposal for the real numbers I will call *the proportionality view of the real numbers*. It claims (remember that 'formal' means being able to be meaningfully ascribed to different kinds of ontological entities):

Real numbers are non-distributive formal proportion-relations between properties.

Using my two examples, and leaving for the moment the term 'proportion-relation' unexplained, it amounts to the following. $\sqrt{2}$ is a proportion-relation between the length of the hypotenuse and the length of the other sides of an isosceles right-angled triangle, be this triangle physical, mental, abstract, or fictional. Similarly, π is a proportion-relation between the length of the circumference and the diameter of a circle, be this circle physical, mental, abstract, or fictional. However, since both $\sqrt{2}$ and π can be proportion-relations between many other pairs of properties, even here Husserl's abstraction into "somethings" is meant to have come into play; this fact is symbolized by the term 'formal' in the expression 'formal proportion-relation'.

As it happens, my views have come to mind when thinking about two of Simons' papers and a very fine description of how physics became arithmetized; see (Simons 2013), (Simons forthcoming), and (Roche 1998, esp. ch. 3). They all take Euclid's treatment of ratios and proportions very seriously.

Even if Simons has never explicitly said so, it is easy to take him to mean that when we say things such as 'the wall is 107 m long', we regard the length of the wall *as if* it consists of a concatenated collection of 107 meter-length units.[22] The determinate length of the wall is in itself only a non-numerical property of the wall, and not a collection, but we can simulate collections in places where there are none. If we allow such *as-if-collections*, then many determinate magnitudes of determinable properties such as length, time duration, mass, and electric charge can be regarded as proportion-relations between two collections. In the case of length, one of the collections is then the collection consisting of the measurement unit for length, and the other is a simulated concatenated collection of such units that covers the magnitude measured.

We are working on the assumption that the natural numbers are properties of collections. Assume now that we have a collection (or as-if-collection) A that has the *property* of having 7 entities and another B that has the *property* of having 8. There are then *relations* between A and B, too. For instance, the sentences 'A has less members than B' and 'B has more members than A' are true. In my very considered opinion, these two sentences are about the same relation; the converse relation predicates 'have less members than' and 'have more members than' refer to the same relation. And the same thing can be said about a number of converse relation predicate pairs: 'smaller-larger', 'colder-warmer', etc. (Johansson 2014b). The view extends to purely arithmetic relations. The once famous mathematician and philosopher of physics Hermann Weyl writes in 1927:

> Two propositions such as '5 follows upon 4' and '4 precedes 5' are expressions of one and the same relation between 4 and 5. It is unwarranted to speak here of two relations inverse to each other. (Weyl 1963, 4)

However, such relations, all of which might be called *more-and-less-relations*, are not what I will focus on; I have mentioned them only in order to show the plausibility of the view that converse relation predicates can describe the same relation. The kind of relations I want to zoom in on, are those that I have already talked about as being *proportion-relations*. Of course, where there is a proportion-relation there is a more-and-less-relation; but not necessarily vice versa.

Take a case where the collection A has to the collection B a proportion-relation of 7 to 8; or, using the converse relation predicate, that B has to A a proportion-relation of 8 to 7. I regard, as said, sentence pairs such as 'A has to B the proportion-relation of 7 to 8' and 'B has to A the proportion-relation of 8 to 7' to describe the same relation. Another way to put it is to say that 'A and B has the ratio 7/8', that 'B and A has the ratio 8/7', and that in these two sentences the two ratio numerals refer to the same proportion-relation.

The same proportion-relation can exist also as a proportion between collections with quite other numbers as relata. The ratio 7/8 is the same as the ratio 49/56, and the proportion-relation between two collections that have the number properties 49 and 56 is the same as the proportion-relation between two collections that have the number properties 7 and 8. If collection A has 7 entities, collection B 8 entities, collection C 49 entities, and collection D 56 entities, then (in Euclid's very terms) A is to B as C is to D. I will call this (second-order) relation between the first-order proportion-relations 'equiproportionality'.[23]

[22] When it is a matter of other properties than that of length, then the term 'concatenation' has to be exchanged for others, but I will disregard all the measurement complications that are hidden here.

[23] What I call 'proportion-relation', Euclid calls 'ratio'; and what I call 'equiproportionality', he calls simply

On the real number line, the ratios 7/8 and 49/56 are placed in exactly the same point. That is, the two ratios correspond to the same rational number.

Just as on the property view of the natural numbers an observed *equinumerosity* is only a criterion for the state of affairs that two collections have the same number of members,[24] on the proportionality view of the real numbers an observed *equiproportionality* is only a criterion for the state of affairs that two pairs of collections have the same proportion-relation.

So far, I have only made it clear that the rational numbers can be regarded as proportion-relations between properties, but the starting point of the investigation was the existence of the irrational numbers. So, what about them? Today, it may seem as if the mathematical constructions that place the irrational numbers on specific places on the real number line, and define them in terms of operations on the rational numbers, are so complicated, that it must be unreasonable to try to extend my view of the rational numbers to that of the irrational numbers. In my opinion, however, this view turns things upside down.

The remark made takes a purely arithmetic realm for granted, more precisely the real number line, and looks then for isomorphic correlations to other realms, but the actual discovery of the irrational numbers went the opposite way. It went from non-arithmetized geometry to arithmetic. It was discovered that the proportion-relations between the length of the hypotenuse (a property) and the length of the sides in an isosceles right-angled triangle (another property), and between the length of the circumference (a property) and the length of the diameter of a circle (another property), cannot be represented by any rational numbers and the then known arithmetical operations. Therefore, in whatever way it is nowadays in pure mathematics possible to define $\sqrt{2}$ and π by means of operations on rational numbers, such constructions cannot show that the irrational numbers do not, like the rational numbers, refer to formal proportion-relations between properties.

One qualification, though, needs to be added; it was triggered by (Michell 1993). He works on a relational account of all numbers, even the natural numbers, and he has noted that proportion-relations are not in themselves necessarily uniquely determined. For instance (my example), the proportion-relation between the circumference and the diameter of a circle is π only in Euclidean geometry; in non-Euclidean geometries it takes on other values. To me, this does not mean that the real numbers cannot be regarded as being proportion-relations between properties; it means that proportion-relations might be unique only in relation to a pre-given theoretical system.

Obviously, it is as meaningful to talk about a proportion-relation between a collection with 2 members and a collection with 1 member, as it is to talk about a proportion-relation between a collection with 7 members and one with 8. But since in pure arithmetic 2/1 = 2, the numeral '2' and all other whole-number-numerals become in the approach I am pursuing ambiguous. When '2' is used as referring to a natural number only among other natural numbers, it is used to refer to a monadic *property* of a collection, but when it is used as referring to a number on the real number line, it cannot refer to a property; it must refer to a *proportion-relation*. The belief that numbers primarily are defined by their place on the number line hides this ambiguity of the whole-number-numerals. When it is not noted, two kinds of mistaken ambiguity-cancelling can occur.

'proportionality'. As I see it, the term 'ratio' has today too strong connotations of necessarily being something arithmetic to fit my purposes; in Euclid's time the term simply meant a kind of relation. Here is a sentence from Roche: "Some early Greek writers on mathematics, however, introduced the concept of the 'number by which a ratio is called'" (Roche 1998, 46).

[24] In his explicit critique of Frege (Husserl 2003, ch. VII), Husserl writes: "The definition of equivalence is, as we have shown, nothing more than a mere criterion for the existence of equality of number in two groups, whereas here [in Frege and some others] it is taken to be a nominal definition of it" (Husserl 2003: 121). Frege had very good reasons to write a long critical review of Husserl's *Philosophy of Arithmetic*. He is explicitly under heavy attack in the book.

One mistake is to think that since the natural numbers are monadic properties, the same must be true of the integer rational numbers, too. The other is to think, conversely, that since all the rational numbers refer to proportion-relations, the same must be true of the natural numbers, too.

The first mistake occurs in lay people who unreflectively talk as if a thing that is, say, 1.75 m long, has this numerical value as an intrinsic property. They do not to themselves make it explicit that to be 1.75 m long is to be 1.75 *times* as long as a standard meter. Normally, this is an innocent mistake that does not affect what they are concerned with. The second mistake occurs in scientists and philosophers who always take the whole number line as their very starting point when thinking about mathematical physics. In the case I will mention, it is not an innocent mistake without consequences.

The International System of Units, the SI-system, is a very widely used system that fixes terminology and standard units for a large variety of measurements in physics and chemistry. Its history dates back to 1875 and the first international meter convention. In this system, the so-called base quantities are treated as if all number value ascriptions in physics and chemistry fit the structure of the real number line. Hereby, the distinction between discrete entities and continuous magnitudes is lost out of sight, which creates unnecessary complications in physical chemistry, where the chemists often use only formulas that contain natural numbers and symbols for count nouns naming kinds of atoms and molecules. In other words, measurement units and proportion-relations are introduced where natural numbers as properties of collections are enough. This criticism is put forward in (De Biévre 2007, 2011, Price & De Biévre 2009, Johansson 2011b, Cooper & Humphry 2012). I think that *one* explanatory factor behind the SI-system's unhappy treatment of discrete entities is the ambiguity of the whole-number-numerals that I have tried to expose.[25]

Before ending the paper, I would like to mention two similarities between the property view of the natural numbers, which brings in the notion of collection, and my proportionality view of the real numbers, which brings in the notion of proportion-relations. As said, the aim of this last section is only to sketch an analysis of the real numbers that has affinity with the property view of the natural numbers.

First, just as a pure natural number is a property abstracted from an entity collection, a pure real number is a proportion-relation abstracted from property pairs. For instance, as 5 is an abstraction from 5 entities, the real number 7/8 is an abstraction from a proportion-relation of 7 to 8.

Second, the property view of numbers is intimately connected with taking the existence of non-distributive plural predication logically seriously. Therefore, it should be noted that also sentences describing proportion-relations contain non-distributive plural predications. For example, in the sentence 'A and B have a proportion-relation of 7 to 8' the term 'A and B' is a plural subject, and the logical predicate 'having a proportion-relation of 7 to 8' is a non-distributive predicate.

***

It would be nice if in the future the property view approach to the ontology of numbers can catch the minds of more philosophers than it has so far. Having said this, I end by thanking Peter Simons for having made at least me aware of this approach.

[25] Two other factors are, I am quite confident, (i) a misplaced longing to have one and the same mathematical structure for all physical-chemical quantities, and (ii) the fact that when scientists have to do with very large ensembles of discrete entities, it becomes mathematically practical for them to use the whole real number line *as if* there could be even an irrational number of discrete entities; (Johansson 2014c, sect. 7).

Acknowledgements I would like to thank Kevin Mulligan, Barry Smith, and Christer Svennerlind for comments on an earlier version of the paper.

References

(All internet references retrieved December 2013)

Armstrong, David M. 1978: *Universals & Scientific Realism I–II*. Cambridge: Cambridge University Press.
—— 2010. *Sketch for a Systematic Metaphysics*. Oxford: Oxford University Press.
Balaguer, Mark 2008: "Fictionalism in the Philosophy of Mathematics". In: Edward N. Zalta (ed.), *Stanford Encyclopedia of Philosophy*. http://plato.stanford.edu/entries/fictionalism-mathematics/. Stanford: The Metaphysics Research Lab.
Bell, David 1990: *Husserl*. London: Routledge.
Ben-Yami, Hanoch 2004: *Logic & Natural Language: On Plural Reference and Its Semantic and Logical Significance*. Aldershot: Ashgate.
Boolos, George 1971: "The Iterative Conception of Set". *The Journal of Philosophy* 68, 215–231.
Bigelow, John 1988: *The Reality of Numbers*. Oxford: Clarendon Press.
Bostock, David 2009: *Philosophy of Mathematics. An Introduction*. Oxford: Wiley-Blackwell.
Campbell, Keith 1990: *Abstract Particulars*. Oxford: Basil Blackwell.
Centrone, Stefania 2010: *Logic and Philosophy of Mathematics in the Early Husserl*. Dordrecht: Springer.
Cooper, Gordon and Humphry, Stephen M. 2012: "The ontological distinction between units and entities". *Synthese* 187, 393–401.
De Bièvre, Paul 2007: "Numerosity *versus* mass". *Accreditation and Quality assurance* 12, 221–222.
—— 2011: "*Integer numbers and their ratios* are key concepts in describing the interactions of atoms and molecules". *Accreditation and Quality assurance* 16,117–120.
Euclid, 300 BC: *Elements*. http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html.
Forster, Thomas 2008: "The Iterative Conception of Set". *The Review of Symbolic Logic* 1, 97–110.
Frege, Gottlob 1972 [1894]: "Review of Dr. E. Husserl's *Philosophy of Arithmetic*". *Mind* 81, 321–337.
Gullberg, Ebba 2011: *Objects and Objectivity. Alternatives to Mathematical Realism*. Umeå: Umeå Studies in Philosophy.
Horsten, Leon 2012: "Philosophy of Mathematics". In: Edward N. Zalta (ed.), *Stanford Encyclopedia of Philosophy*. http://plato.stanford.edu/entries/philosophy-mathematics/. Stanford: The Metaphysics Research Lab.
Hume, David 2000: "A Treatise of Human Nature". In: D.F. Norton and J.F. Norton (eds.), *David Hume: A Treatise of Human Nature*. Oxford: Oxford University Press, 1–417.
Husserl, Edmund 2003 [1891]: *Philosophy of Arithmetic*. Dordrecht: Kluwer Academic Publishers.
—— 1970: *Logical Investigations* vols. I [1913] and II [1921): London: Routledge & Kegan Paul.
—— 1982 [1913]: *Ideas pertaining to a pure phenomenology and to a phenomenological philosophy (first book)*: Dordrecht: Kluwer Academic Publishers.
Johansson, Ingvar 2002: "Determinables as Universals". *The Monist* 83, 101–121.
—— 2004: *Ontological Investigations*. Frankfurt: Ontos Verlag.
—— 2010: "Fictions and the Spatiotemporal World—in the Light of Ingarden". *Polish Journal of Philosophy* IV, 81–103.
—— 2011a: "John Searle in the year 2010—Reviews". *Metaphysica* 12, 73–85.
—— 2011b: "The Mole is Not an Ordinary Measurement Unit". *Accreditation and Quality assurance* 16, 467–470.
—— 2012: "Hume's Ontology". *Metaphysica* 13, 87–105.
—— 2013: "The Ideal as Real and as Purely Intentional – Ingarden-based Reflections". *Semiotica* 194, 21–37.
—— 2014a: "Proof of the Existence of Universals—from a Fallibilist". In: G. Bonino & G. Jesson (eds.), *Defending Realism*. Berlin: De Gruyter, 51–61.
—— 2014b: "On Converse Relations". In: H. Malmgren & T. Nordin (eds.), *Philosophy and Botany*. Stockholm: Thales, 82–106.
—— 2014c: "Constancy and Circularity in the SI". In: MetrologyBytes.net: http://www.metrologybytes.net/opEds2014.php
Körner, Stephan 1960: *The Philosophy of Mathematics. An Introductory Essay*. London: Hutchinson.
Lowe, E.J. 2006: *The Four-Category Ontology*. Oxford: Clarendon Press.
McCrink, Koleen and Spelke, Elizabeth S. 2010: "Core Multiplication in Childhood". *Cognition* 116, 204–216.
McKay, Thomas J. 2006: *Plural Predication*. Oxford: Clarendon Press.

Michell, Joel 1993: "Numbers, Ratios, and Structural Relations". *Australasian Journal of Philosophy* 71, 325–332.
Mulligan, Kevin 2013: "Formal Concepts". In: K. Mulligan, K. Kijania-Placek & T. Placek (eds.). *The History and Philosophy of Polish Logic*. Basingstoke: Palgrave Macmillan, 205–223.
Oliver, Alex and Smiley, Timothy 2013: *Plural Logic*. Oxford: Oxford University Press.
Park, Jee-Hyun and Nunes, Terezinha 2001: "The Development of the Concept of Multiplication". *Cognitive Development* 16, 763–773.
Price, Gary and De Bièvre, Paul 2009: "Simple principles for metrology in chemistry: identifying and counting". *Accreditation and Quality assurance* 14, 295–305.
Roche, John C. 1998: *The Mathematics of Measurement: A Critical History*. London: The Athlone Press.
Russell, Bertrand 2006 [1903]: *Principles of Mathematics*. London: Routledge.
Searle, John 1995: *The Construction of Social Reality*. New York: The Free Press.
—— 2010: *Making the Social World*. Oxford: Oxford University Press.
Simons, Peter 1982a: "Number and Manifolds". In: B. Smith (ed.), *Parts and Moments*. Munich: Philosophia, 160–198.
—— 1982b: "Plural Reference and Set Theory". Ibid., 199–256.
—— 1987: "Frege's Theory of Real Numbers". *History and Philosophy of Logic* 8 (1987), 25–44. Reprinted in P.M. Simons, *Philosophy and Logic in Central Europe from Bolzano to Tarski*. Dordrecht: Kluwer, 1993, 117–142, and in W. Demopoulos (ed.), *Frege's Philosophy of Mathematics*. Cambridge, MA: Harvard University Press, 1995, 358–385.
—— 1991: "Faces, Boundaries, and Thin Layers". In: A.P. Martinich & M.J. White (eds.), *Certainty and Surface in Epistemology and Philosophical Method. Essays in Honor of Avrum Stroll*. Lewiston: Edwin Mellen Press, 87–99.
—— 1994: "Particulars in Particular Clothing: Three Trope Theories of Substance". *Philosophy and Phenomenological Research* 54, 553–575.
—— 2005: "Against Set Theory". In: J.C. Marek & M.E. Reicher (eds.), *Erfahrung und Analyse/Experience and Analysis. Proceedings of the 2004 Wittgenstein Symposium*. Vienna: öbv&hpt, 143–152.
—— 2007: "What Numbers Really Are". In: R.A. Auxier & L.E. Hahn (eds.), *The Philosophy of Michael Dummett*. Chicago: Open Court, 229–247.
—— 2009: "Why There Are No States of Affairs". In: M.E. Reicher (ed.), *States of Affairs*. Frankfurt: Ontos Verlag, 111–128.
—— 2010: "Foreword". In: (Centrone 2010, v–viii).
—— 2011: "On Multitudes". Lecture at Opole University: http://www.glli.uni.opole.pl/publications/LT_2011_10_en.pdf
—— 2013a: "Density, Angle, and other Dimensional Nonsense: How Not to Standardize Quantity". In: C. Svennerlind, J. Almäng, & R. Inghtorsson (eds.), *Johanssonian Investigations*. Frankfurt: Ontos Verlag, 516–534.
—— 2013b: "Arithmetic in Leśniewski's Ontology". In: K. Mulligan, K. Kijania-Placek, & T. Placek (eds.). *The History and Philosophy of Polish Logic*. Basingstoke: Palgrave Macmillan, 227–241.
—— forthcoming: "Euclid's Context Principle". *Hermathena – A Trinity College Dublin Review* (a Festschrift issue for Prof. John Cleary, 1949–2009).
Smith, Barry 2000: "Truth and the Visual Field". In: J. Petitot, F.J. Varela, B. Pachoud, & J.M. Roy (eds.), *Naturalizing Phenomenology. Issues in Contemporary Phenomenology and Cognitive* Science. Stanford: Stanford University Press, 317–329.
Smith, Barry, and Varzi, Achille 2000: "Fiat and Bona Fide Boundaries". *Philosophy and Phenomenological Research* 60, 401–420.
Varzi, Achille 2008: "Boundary". In: Edward N. Zalta (ed.), *Stanford Encyclopedia of Philosophy*. http://plato.stanford.edu/entries/boundary/. Stanford: The Metaphysics Research Lab.
Weyl, Hermann 1963 [1927]: *Philosophy of Mathematics and Natural Science*. New York: Atheneum.
Willard, Dallas 2003: "Translator's Introduction". In: (Husserl 2003, xiii–lxiv).
Yi, Byeong-Uk 1998: "Numbers and Relations". *Erkenntnis* 49, 93–113.
Yoshida, Makoto 2009: "Is Multiplication Just Repeated Addition?" Lecture at 2009 NCTM Annual Conference: http://www.globaledresources.com/resources/assets/042309_Multiplication_v2.pdf .