Mathematical Vectors and Physical Vectors

Ingvar Johansson[†]

Pre-print version; the paper is published in *dialectica* Vol. 63, N° 4 (2009), pp. 433–447.
DOI: 10.1111/j.1746-8361.2009.01215.x.
The published version is available at www.blackwell-synergy.com.

Abstract

From a metaphysical point of view, it is important clearly to see the ontological difference between what is studied in mathematics and mathematical physics, respectively. In this respect, the paper is concerned with the vectors of classical physics. Vectors have both a scalar magnitude and a direction, and it is argued that neither conventionalism nor whole-sale anti-conventionalism holds true of either of these components of classical physical vectors. A quantification of a physical dimension requires the discovery of ontological order relations among all the determinate properties of this dimension, as well as a conventional definition that connects the number one and (in case of vector quantities) mathematical unit vectors to determinate spatiotemporal physical entities. One might say that mathematics deals with *numbers* and *vectors*, but mathematical physics with *scalar quantities* and *vector quantities*, respectively. The International System of Units (SI) distinguishes between basic and derived scalar quantities; if a similar distinction should be introduced for the vector quantities of classical physics, then duration in directed time ought to be chosen as the basic vector quantity. The metaphysics of physical vectors is intimately connected with the metaphysics of time. From a philosophical-historical point of view, the paper revives W. E. Johnson's distinction 'determinates-determinables' and Hans Reichenbach's notion of 'coordinative definition'.

For many centuries, basic physics has meant mathematical physics. This does not mean that physics has merged with mathematics, or vice versa. Just think of arithmetic. In the spatiotemporal world there are no pure mathematical numbers. Of course, we can find things such as 1 pebble, 2 pebbles, 3 pebbles, etc., 1 chair, 2 chairs, 3 chairs, etc., a thing of 1 m, a thing of 2 m, a thing of 3 m, etc., a thing of 1 kg, a thing of 2 kg, a thing of 3 kg, etc. And so on with respect to both kinds of things and kinds of physical dimensions. By means of the numerals '1', '2', '3' in isolation, however, we cannot refer to any numbers 1, 2, 3 that are situated in space and time. Before arithmetic can enter the discipline of physics, numerals

[†] Professor emeritus in theoretical philosophy, Umeå University, Sweden; Virvelvindsvägen 4P, SE-222 27 Lund, Sweden; Email: ingvar.johansson@philos.umu.se

have to be connected to count nouns or something that functions as count nouns. Out of a mass noun such as ‘water’ we can create count nouns such as ‘glass of water’ and ‘water molecule’ and then speak of numbers of both glasses of water and water molecules. Also, given that certain conditions are fulfilled, physical dimensions such as length and mass can be ascribed a measurement unit that functions as a count noun, e.g., ‘1 m’ and ‘1 kg’. Pure mathematics deals with *numbers*, but mathematical physics deals with *quantities*. Whatever the true ontological status of different kinds of numbers is, such numbers cannot have any direct instances or exemplifications in the physical world. Surely, pure numbers are abstract in the sense of not being spatiotemporally localizable, whereas quantities are meant to be able sometimes to be spatiotemporally localizable. In Section 1 below, it is shown that this ontological difference between the referents of mathematics and mathematical physics explains why mathematical physics are using what Hans Reichenbach (1958, §4) calls ‘coordinative definitions’.

From any vector quantity a non-directed scalar quantity can be abstracted, and my analysis of physical scalar quantities in Section 1 applies to the scalar component of vectors, too. Then, in Section 2, I will analyze the directedness that is specific to physical vectors; the result being that even such directedness is partly non-conventional (i.e., is ontologically grounded) and requires coordinative definitions, at least if time is assumed to have a direction.

Mainly, I will stay within the confines of classical physics (Newtonian mechanics, classical thermodynamics, and Maxwellian electromagnetic theory). In order ever to become clear about the metaphysics of the vectors of quantum mechanics, the tensors of general relativity, and the quantities of string theory. I think we first have to see the ontological commitments of the vectors of classical physics at least somewhat more clearly.

1. Pure mathematical numbers and physical scalar quantities

Scalar quantity expressions such as ‘1 meter’, ‘2 kilogram’, and ‘3 kelvin’ contain a *numeral* connected to a *name of a measurement unit* (‘meter’, ‘kilogram’, ‘kelvin’). Similarly, expressions such as ‘1 m’, ‘2 kg’, and ‘3 K’ contain a numeral connected to a *symbol* of a measurement unit. All these expressions have only two parts, but implicitly there is a third part involved. This fact comes out clearly when one looks at statements that are used to report results of measurements. Here are some examples related to an object/body called ‘b’: ‘b is *1 m long*’, ‘b has a *mass of 2 kg*’, and ‘b has a *temperature of 3 K*’. The third part is a *name of* *a physical dimension*; in the cases at hand the dimensions are length, mass, and temperature, respectively. Complete scalar quantity expressions (Q_S) such as the last ones conform to an abstract schema that has at least the following three parts (the dots in the schema have only the function of separating the parts distinguished; they have no mathematical or traditional linguistic meaning):

- Q_S = (numeral) · (name or symbol of measurement unit) · (name of physical scalar dimension)

Bringing in traditional philosophical terms, and using not only the formal mode of speech (expressions, numerals, names, symbols) but also the material mode (quantities, numbers, measurement units, physical dimensions), this means that scalar quantities are presupposing W. E. Johnson's (1964a, chap. XI) distinction between *determinates* and *determinables*.[1] Each physical dimension is a determinable, and each numeral refers with the help of a name/symbol of a measurement unit to a certain determinate of the determinable or physical dimension named; in what follows, I will use the expressions 'physical dimension' and 'physical determinable' interchangeably. If the numerals in the expressions '1 m length', '2 kg mass', and '3 K temperature' were not connected to symbols of a measurement unit and a name of a determinable, they would only refer to pure mathematical numbers; and if the names of the determinables ('length', 'mass', 'temperature') were not connected with numerals and symbols of measurement units, they could not be used to refer to anything determinate in the spatiotemporal world.

Now, as everyone knows, the determinable length can be ascribed not only the measurement unit '1 meter' but also that of others such as '1 yard'; similarly the determinable mass has been ascribed not only the measurement unit '1 kilogram' but also (in England) that of '1 slug' (= 14.5939 kg), the determinable temperature is outside of basic physics also ascribed degrees of Celsius and Fahrenheit; and so on for all possible physical dimensions. This obvious (i) *conventionality of the measurement unit* of a physical determinable must not be interpreted as implying also (ii) a *conventionality with regard to how physical determinates*

[1] For an overview of this distinction and its problems, see (Sanford, 2006). My own views are expounded in more detail in (Johansson, 2000). W. E. Johnson has the credit of being the philosopher who first tried to make the distinction philosophically significant in twentieth century philosophy; the book (Johnson, 1964a) was first published 1921. However, as he himself remarks, a determinable "is almost equivalent to" what Descartes and Spinoza call an attribute (Johnson, 1964c, p. 67, n1).

are related to physical determinables. On the contrary, even though measurement units are conventional, it is not a matter of convention whether or not a set of different determinates can constitute a physical determinable. What this combined conventionality (of measurement unit) and non-conventionality (of determinates-to-determinable relation) consists in, I will now explain. Let me just first say that it means that we do not directly in nature meet any quantities, but that nonetheless there is something in nature that can be made to correspond to quantities (Johansson, 2008). The non-conventionality, which is ontological in nature, can be shown as follows.

Scalar quantities of a certain physical dimension can be linearly ordered in the same way as the pure mathematical numbers are ordered on the number line. Therefore, just as there is an asymmetric (and transitive) relation larger-than that obtains between all the different pure mathematical numbers, there has to be an asymmetric relation that obtains between all the determinates of the physical dimension in question. In the case of length we have the asymmetric relation longer-than, in the case of mass larger-than, and in the case of temperature higher-than; in the case of shape we have so far not found any such relation, i.e., shape has not yet been quantified.[2] But *it is only determinates of the same determinable that can be ordered by such relations*; a fact that I will call 'the basic ontological requirement of physical quantification'. For instance, 5 kg is more than (larger-than) 3 kg, but it is neither more nor less than 3 m or 3 K. Determinates of different physical dimensions are incommensurable.

Furthermore, when scalar quantities are additive, *they can in a physically meaningful way only be added to (or subtracted from) other determinates of the same determinable*; call this 'the law of addition and subtraction of quantities' (Johansson, 2000, 116). 5 kg + 3 kg equals 8 kg, and 5 m + 3 m equals 8 m, but 5 kg + 3 m equals nothing at all.[3] It must be the 'nature' of the determinates in question, not pragmatic conventions, that decide whether or not a collection of determinates can be said to belong to the same physical dimension. For the purposes of this paper, the 'natures' spoken of can be regarded either as universals (realism) or as equivalence classes of property instances or tropes based on a similarity relation (resemblance nominalism).

[2] In my paper (Johansson, 2008, sec. 3), I argue that this is not by accident. The shape determinable cannot be quantified since there are more possible determinate shapes than there are real numbers, i.e., a one-to-one association between the shape determinates and the real numbers is logically impossible.

[3] I find it remarkable that this 'law' and the just mentioned 'ontological requirement' are absent from much contemporary philosophy of science; see for instance my criticism of E. J. Lowe in (Johansson, 2006).

In the sense explained, every physical scale contains the phenomenon of identity-in-difference, i.e., one-determinable-in-different-determinates. If physics is regarded as more than a purely conceptual game, then each *most specific* determinate has to be regarded as having some kind of nature independently of language. This fact is also underlined by the following truth: no physical object can simultaneously exemplify two determinates of the same physical determinable; call this 'the principle of determinate exclusion' (Johansson, 2000, 116; Johnson, 1964a, 181 and 237; 1964b, 149 and 195). For instance, nothing can have two determinate masses and be represented by two numerical values that are connected to the same measurement unit.

Ever since Frege's and Russell's criticism of Kant's analysis of mathematics, it has been debated whether pure mathematical truths are synthetic a priori or analytic a priori truths. Be this for this paper's sake as it may. But it seems to me as if what I have called 'the basic ontological requirement of physical quantification', 'the law of addition and subtraction of quantities', and 'the principle of determinate exclusion' may lay claim to have more or less the same kind of epistemological necessity as mathematical truths have, whatever that is.[4] The laws mentioned cannot be empirically tested. It is in quite a strong sense I think that it is not a matter of convention whether or not a class of determinates can constitute a *physical* determinable. Of course, one can create purely nominalistic sets such as 'the set of all length determinates and all mass determinates' and 'the set of what is 3 m, what is 5 kg, and what is 200 K', but such sets are not determinables in the original identity-in-difference sense; and such sets will never be able to function in a physical law (Johansson, 2001, 109-113).

This non-conventionality claim is an ontological claim about *determinates*-in-relation-to-determinables. It is not concerned with the issue about the conventionality or non-conventionality of *determinables*-in-relation-to-determinables. This issue is of relevance for the philosophical interpretation of The International System of Units (abbreviated SI). The SI system was created in 1960, and has since then been the authoritative system of measurement units within the natural sciences. Since 1971, it contains seven undefined quantitative physical dimensions, so-called 'base quantities', e.g., length, mass, time, and thermodynamic temperature. All other quantities in physics are called 'derived quantities', since they by means of multiplication and/or division are defined or characterized by a specific relation that they have to the base quantities. Sometimes this relation looks like an a priori truth, as in

[4] In passing, I will mention that I am convinced that a similar view is defended by Edmund Husserl; see (Smith, 1994, pp. 305-311) and (Johansson, 2004, chap. 9.2).

‘(velocity) $=_{def}$ (length) x $(time)^{-1}$’, but often it is based on a classic natural law as, for instance, in ‘(force) $=_{def}$ (mass) x (length) x $(time)^{-2}$’, and ‘(energy) $=_{def}$ (mass) x $(length)^2$ x $(time)^{-2}$’. Every base quantity is also ascribed a conventional *base unit*; length has meter, mass has kilogram, time second, and temperature kelvin. The metaphysically relevant question is whether it is merely a matter of pragmatic convention what determinables should be regarded as the undefined basic ones.

From a purely *physical calculation point of view*, which is that adopted in the SI system, one need not necessarily define velocity by means of length and time; length can be turned into a defined quantity and velocity into an undefined one by means of the definition ‘(length) $=_{def}$ (velocity) x (time)’. Such a definition, however, does not take the ontology of physics into account. Assume that some physical determinables are for their existence *one-sidedly* existentially dependent on other determinables (Johansson, 2004, chap. 9.3; Lowe, 2005, sec. 3), i.e., the former cannot be physically exemplified if the latter are not exemplified, whereas the latter might be exemplified in isolation from the former. When this is the case, the latter ought from an *ontological point of view* to be regarded as undefined basic determinables, and the former as defined and derived determinables. It would be ontologically odd to make the referents of an undefined term existentially dependent on the referents of some defined terms.[5] This line of thought can be exemplified as follows. If Newton’s mechanics, with its so to speak natural ontological commitments, were to be regarded as true, then velocity should from an ontological point of view be defined in terms of length and time, not length in terms of velocity and time. Why? Because this theory allows space and time to be empty, but not particles (be they moving or not) to exist in isolation from space and time. That is, velocity is in this theory one-sidedly existentially dependent on length and time.

If there are relations of one-sided existential dependence between determinables, then these relations are non-conventional relations between determinables and determinables. This means, that whereas from a purely physical-calculation point of view the relationships between determinables can be regarded as conventional, from an ontological-dependency point of view this is not always the case. At the end of Section 2, I will return to this fact. Here, however it should just be noted and then put aside; this is in order not to mess it up with either the non-conventionality of the physical *determinate*-to-determinable relation or the conventionality of measurement units. Next follows comments on the latter conventionality.

[5] Note that I am leaving epistemic dependence relations wholly out of account.

A necessary requirement for a quantification of a physical dimension is that the latter's determinates allow themselves to be linearly ordered. But in itself such an ordering creates only a so-called 'ordinal scale'. In order to obtain quantities, also relations between *equality of intervals* in the ordinal scale have to be found; for instance, the interval between the determinates that after the quantification is referred to by 1 kg and 2 kg should be exactly the same as the interval between the determinates later referred to by 1026 and 1027 kg. If there are such intervals, then a so-called 'interval scale' (such as the Celsius and Fahrenheit scales) can be construed; when some further requirements are met a 'ratio scale' (such as the ones for length and mass) can be construed. However, the distinction between interval scales and ratio scales (Kaplan, 1964, §22; Hand, 2004, chap. 2.2.5) need not detain us here. What is of crucial importance, however, is that in order to obtain a 'metric scale' and quantities, a *coordinative definition* (Reichenbach, 1958, §4) is needed.

A coordinative definition is neither a definition that stipulates or reports that a certain word meaning is equivalent to some other word meanings (classic example: 'bachelor $=_{def}$ unmarried man'), nor an Aristotelian real definition that lays claim to have found a necessary mutual relationship between some universals (classic example: 'man $=_{def}$ rational animal'), i.e., it is neither a relation between words or meanings, nor a relation between universals (or, for that matter, equivalence classes of property instances or tropes). A coordinative definition *creates by stipulation* a relation between, on the on the hand, a linguistic entity such as a term, a name, an expression, a meaning, or a concept, and, on the other hand, something extra-linguistic such as a property of an individual thing, a universal, an equivalence class of property instances or tropes, or to some of these kinds of entities simultaneously.[6]

On the linguistic assumption that for many terms it holds true that the meaning of the term determines its extension, one might say that coordinative definitions are needed for all categorematic terms where this is not the case.[7] In my opinion, the basic deictic words 'I', 'here', and 'now', have implicit coordinative definitions that say that 'I' refers to the speaker, 'here' refers to the speakers spatial position, and 'now' refers to the speakers temporal position. One might even say that it is part of the meaning of deictic words that they cannot refer only by means of their meaning. And the same goes for the meaning of terms for

[6] If one thinks there are both universals and tropes in the spatiotemporal world, as moderate realists do, one can consistently claim that a coordinative definition simultaneously relates a term to both a universal and one of its instances/tropes.

[7] A categorematic term is a word with semantic content; syncategorematic terms such as the logical constants have so to speak only syntactic content.

measurement units such as '1 m', '1 yard', '1 kg', etc. It is part of their meaning that somewhere there is an explicit coordinative definition that stipulates, directly or indirectly, a connection between the term and a determinate physical property.

In the SI system there are at the moment two different kinds of metric coordinative definitions. The expression '1 kg mass' is by a stipulation, collectively accepted by the natural-scientific community, made to refer to the determinate mass of a certain solid platinium-iridum cylinder, 'the standard mass', that is kept in Paris. That is, this metric coordinative definition relates the expression '1 kg mass' directly to a particular property instance or trope; which, because of various theoretical considerations, is assumed to retain its identity in the foreseeable future. Then, of course, all tropes that are exactly similar to this prototypical trope should be regarded as having the mass of 1 kg, too.[8] If realism with respect to universals is accepted, the standard mass prototype contains an instance of a determinate universal of the determinable mass, and all other instances of this universal have necessarily the mass of 1 kg, too.

The expression '1 m length', on the other hand, is not directly related to one particular property instance or trope. It becomes connected with the spatiotemporal world in an indirect way. To start with, its extension is by stipulation made identical with the class of instances that fall under the description "the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second" (SI, 2006, 112). Once there was a prototypical 'standard meter' in Paris, but in 1983 it was exchanged for the definition just presented. This definition, however, contains a reference to still another base unit, that of '1 second'. If one wants a functioning coordinative definition, then such a measurement unit regress cannot be allowed to go on indefinitely. And it doesn't. The SI definition of '1 second' says: "The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom" (SI, 2006, 113). Here, there is no reference to still another base unit, which means that the term 'period' is implicitly taken to function as a primitive count noun. Therefore, the definition of '1 s' is a rock-bottom coordinative definition, upon which the coordinative definition of '1 m' rests.

[8] I will not at all discuss the huge epistemological problems that are connected with how to determine when a mass trope is exactly similar to the standard mass in Paris. This paper is only concerned with ontology and semantics.

The view I have put forward implies that the scalar variables of classical physics are not variables for *numbers*, but variables for *scalar quantities* that represent linearly ordered physical determinates of various determinables. In Galilei's law for falling bodies, $s = ½gt^2$, s is a variable for length quantities, and t is a variable for duration quantities; in the scalar formulation of Newton's second law, $F = ma$, F is a variable for the numerical representations of the determinates of the determinable force, m for mass, and a for acceleration, i.e., for force quantities, mass quantities, and acceleration quantities, respectively.

In a singular statement such as 'b has a mass of 2 kg', the logical predicate ('mass of 2 kg') is neither a pure monadic predicate (such as F in 'Fb') nor a pure relation predicate (such as R in 'Rbc'), but a predicate that refers to a so-called *relational property*. It might be called a 'relation*al* predicate', and be formalized as '(Rc)b'. The statement says that the body b has twice the mass of the standard measurement unit, c. But this is quite consistent with the fact that b has in itself, quite independently of the standard unit, a monadic property, a certain determinate mass; a relational predicate can refer to a non-relational property. Both b and the standard unit for mass (i) have in themselves a monadic determinate mass, and between these determinate monadic masses (ii) there is a two-term determinate difference-of-mass relation; and the relation*al* predicate is able simultaneously to take account of both (i) and (ii).

A metric scale can be regarded as an ingenious linguistic device in which the use of a relational predicate such as '2 kg' can simultaneously bring with it references to both a monadic property (the determinate mass of b) and to a relational property (twice the mass of the standard unit). But more than that; implicitly, it also brings in relations to all other masses that are ascribed a determinate mass by means of expressions of the form 'x kg'. All entities that are directly related to the same standard unit are thereby also indirectly related to each other. The *order* among them must by no means be conflated with the *directedness* that vectors and vector quantities display.

2. Pure mathematical vector directedness and physical vector directedness

There are two main definitions of 'vector' around in mathematics. The one that I will fasten on to says:

- **vector** An entity in Euclidean space that has both magnitude and direction (*Dictionary of Mathematics*, 2003, 440).

Many mathematicians, however, prefer another and more formal definition that superficially takes the 'direction' away. I would like to present this definition as consisting of three recursive parts:

(i) vector $=_{def}$ element in a vector space

(ii) vector space $=_{def}$ abstract space for which two operations on its elements are defined: addition and multiplication of scalar

(iii) element (of an n-dimensional vector space) $=_{def}$ mathematical entity that is represented by an ordered list of n real numbers.

In this definition, the term 'an ordered list of n real numbers' implicitly brings back the 'direction' that was explicitly mentioned in the first definition; the order spoken of has to have a direction. Therefore, in what follows, I will take it for granted that it makes good sense to speak of a *mathematical directedness* as being one component of mathematical vectors.

Now, there are not only pure mathematical vectors, there are physical vectors, too. Therefore, two different vector characterizations are needed: (a) a *vector* is a pure mathematical entity that has a scalar magnitude that is equal to a number as well as a direction in an abstract space; (ii) a *vector quantity* is a quantity that has a scalar magnitude that is equal to a scalar quantity as well as a direction in physical space or along a physical dimension.

Let me now spell this view out in some detail.[9] The fundamental vector quantities in classical physics are: velocity, momentum, acceleration, force, electric current, electric field strength, and magnetic field strength. To begin with, I will use only velocity as an example of a vector quantity, but I will end with the claim that none of the quantities mentioned would have directedness if time were not regarded as having a direction.[10] The view that time is directed and always flows forward seems to have been a common ontological commitment

[9] What follows is to some extent based on (Johansson, 2001 and 2008), but, for instance, coordinative definitions are not mentioned in either of these papers.

[10] Note, by the way, that circular time, just as well as linear, can be ascribed a direction.

among the classical physicists, even though it was not explicitly built into any law before the second law of thermodynamics was stated.[11]

In the analysis of scalar quantities, it was necessary to oscillate between the formal and the material mode of speech; the reason being that there is no completely non-conventional relation between quantity expressions and the determinates that they can be used to refer to. But even in this section such an oscillation will be useful. There is an abstract schema for vector quantity expressions (Q_V) that corresponds to that presented for scalar quantity expressions (Q_S). Here come both:

- Q_S = (numeral) · (name/symbol of measurement unit) · (name of physical scalar dimension)
- Q_V = (vector expression) · (name/symbol of measurement unit) · (name of physical vector dimension)

The directedness of vector quantities does always relate to a physical dimension; if one determinate of a determinable is directed, then so are all the other determinates.[12]

Let me now start to talk about the physical vector dimension of *velocity* and the physical scalar dimension of *speed*. However, because of a certain instrumentalism-versus-realism problem, I have to make a detour before I can apply the schemas above to speed and velocity.[13]

In classical physics, velocities are velocities of particles or fields. And, one might like to add, velocities of *moving* particles and fields. But here one must be cautious. In the fundamental laws, the velocity variable is a variable for *momentary* velocities, but in an absolutely momentary point of time there is by definition no movement. The solution to this problem can be searched along two opposed lines. Taking an instrumentalist stance, one can try to argue that momentary velocity should be treated only *as if* it exists; from a purely mathematical-calculation point of view there is no problem. Taking a realist stance, one can try to argue that a particle that in a momentary point of time has a velocity differs from one

[11] In Newton's laws, Maxwell's equations, and all the laws of thermodynamics except the second one, it makes no difference if the variable for time, 't', is exchanged for '-t'. But if such a change is made in the second law of thermodynamics, the expression '*in*creasing entropy' must be exchanged for that of '*de*creasing entropy'.

[12] Null vectors are here and in what follows disregarded.

[13] In what follows, the 'S' in 'Q_S' can symbolize both 'scalar' and 'speed', and the 'V' in 'Q_V' both 'vector' and 'velocity'.

that has not in the following way: the particle with $v \neq 0$ has a 'potentiality for movement' or 'tendency to movement' that the particle with $v = 0$ lacks. In what follows, I will sometimes without qualifications simply write as if velocity is always velocity of an actual movement.[14]

The particles and fields of classical physics are moving in three-dimensional space. That is, the directedness of velocity, which is the directedness of a movement, is directedness in physical space; it is spatial directedness. Therefore, with respect to vectors, let us look at a three-dimensional abstract vector space. I will use the vector notation '$\mathbf{e} = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$', where $\mathbf{e}$ is an arbitrary vector, and the unit vectors $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$ are orthogonal to each other.

Every velocity vector quantity ($\mathbf{v}$) has the form '$\mathbf{v}$ m/s = $(v_1\mathbf{e_1} + v_2\mathbf{e_2} + v_3\mathbf{e_3})$ m/s'. The corresponding *speed* scalar quantity (s) has the form 's m/s = $(\sqrt{}v_1^2 + v_2^2 + v_3^2)$ m/s'. When a vector is turned into a scalar (number) by the displayed transformation rule, the measurement unit stays the same. But what does the relation between the physical dimensions of velocity and speed look like? Let us apply the abstract schemas Q_S and Q_V to the case at hand. With respect to speed we get:

- $Q_S = (\sqrt{}v_1^2 + v_2^2 + v_3^2) \cdot (\text{m/s}) \cdot (\text{speed})$

The corresponding schema for velocity must (implicitly or explicitly) contain a name also for a dimension of physical directedness, and the schema looks like this:

- $Q_V = (v_1\mathbf{e_1} + v_2\mathbf{e_2} + v_3\mathbf{e_3}) \cdot (\text{m/s}) \cdot (\text{velocity})$
  $= (v_1\mathbf{e_1} + v_2\mathbf{e_2} + v_3\mathbf{e_3}) \cdot (\text{m/s}) \cdot (\text{speed}) \cdot (\text{spatial directedness})$

When the last expression is turned into the scalar expression of speed, the name of the physical dimension of spatial directedness simply disappears. This is of course quite in conformity with the fact that all the unit vector expressions, '$\mathbf{e_1}$', '$\mathbf{e_2}$', and '$\mathbf{e_3}$', disappear, too.

In Section 1, I presented my answer to the question what the link between the number 1 and scalar quantities such as 1 m and 1 kg consists in. Now we meet the question what the link between the mathematical unit vectors ($\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$) and physical spatial directedness looks

[14] The point now made about 'momentary velocity' can also be made about the more general phenomenon 'momentary rate of change'. Velocity can be defined as 'rate of change of spatial position' or 'intensity of displacement/movement'.

like. This time we can get no clues from the SI system, since it is concerned with scalar quantities only.

A mathematical vector may be, and sometimes is, defined as ‘a directed line segment’ (Pettofrezzo, 1994, 2); a unit vector has the magnitude one along a straight line in a certain direction. If one stays (as I will) within Newtonian mechanics and three-dimensional Euclidean geometry, one might for simplicity’s sake and the purpose of this paper say that the meanings of the terms ‘mathematical straight line’ and ‘physical straight line’ determine their extensions; and that, therefore, there is in the present context no need for any coordinative definition of ‘straight line’. If, however, the theory of general relativity were to be brought in, then, as the philosophy of general relativity has amply shown, a coordinative definition of ‘straight line’ would be necessary; in fact, there seems to be a consensus to the effect that the path of a ray of light in vacuum should be considered such a ‘standard straight line’. This my remark about ‘straight line’ applies equally well to ‘orthogonal lines’. I will treat also this expression as semantically unproblematic, but if the constructions of non-Euclidean geometries and the theory of general relativity were to be taken into account, neither would this be allowable.

These remarks being made, it is nonetheless the case that the connected unit vectors ($\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$) need a coordinative definition. Since in Newtonian space a line always contains two possible directions, a direction along such a line can only be established by a conventional coordinative definition. In a quantification of a scalar, a measurement unit such as ‘1 kg’ has to be chosen by convention. In Newtonian mechanics, analogously, spatial directedness has to be chosen by convention. The magnitude of the unit vectors, 1, is taken care of by the measurement unit of length, which I have already discussed.

Let me now exploit the fact that there are two common symbols used for vectors, and tie one symbol, ‘($\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$)’, to the vectors of mathematics and the other, ‘($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$)’, to the vector quantities of physics. The numeral ‘1’ can start to function in relation to a name of a physical dimension as soon as there is an adequately constructed measurement unit. Similarly, a unit vector expression ‘($\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$)’ can start to function in physics as soon as there are three orthogonal directed lines in physical space, ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$), to which they are coordinated ($\mathbf{e_1} \rightarrow \mathbf{i}$, $\mathbf{e_2} \rightarrow \mathbf{j}$, and $\mathbf{e_3} \rightarrow \mathbf{k}$). When such a coordinative definition exists, we can write as follows:

- $Q_V = (v_1\mathbf{e_1} + v_2\mathbf{e_2} + v_3\mathbf{e_3}) \cdot (\text{m/s}) \cdot (\text{speed}) \cdot (\mathbf{i}, \mathbf{j}, \mathbf{k})$

The conventionality of the directions of **i**, **j**, and **k** is a conventionality against a background of some presupposed hard non-conventional ontological facts. The coordinative definition has to presuppose that space in itself is three-dimensional. Now to be noted is: even though Newtonian space in itself contains no directedness, only human-imposed directedness, a velocity has in itself spatial directedness. This direction stays the same independently of whether it is related to the frame ($\mathbf{i}^1$, $\mathbf{j}^1$, $\mathbf{k}^1$) or to another frame ($\mathbf{i}^2$, $\mathbf{j}^2$, $\mathbf{k}^2$). This is no more curious than the fact that the statements 'b is 1 m' and 'b is 1.0936133 yards' describe exactly the same length.

Velocity is always a velocity of a movement, and the velocity inherits its spatial directedness from the spatial directedness of the movement. A non-momentary movement is extended in time and is an actual movement between spatial positions; a momentary movement is only potentially a movement,[15] and it has only a potential directedness towards other spatial positions. Movement brings in two ordered physical dimensions, a spatial order and a temporal order; a change of spatial position does, just like any change, require both the order of what is changing (spatial positions or determinates of some physical dimension) and the time order. An order, however, as should be clear from Section 1, is something other than directedness. If neither space nor time has direction, then it makes no sense to speak of any directions of movements either. A movement is directed because it is *first* at one spatial position and *later* (actually or potentially) at another position. The directedness of a movement is a *derived* directedness; it is founded on the direction of time, not just on the order between time points.

A movement is *not*, note, a movement between *temporal positions*, even though the existence of the movement depends on the existence of directed time; a movement is a movement between (a change of) *spatial positions* that *takes place in directed time*.

These remarks about change of spatial position fit property changes, too. Suppose, for instance, that a rod undergoes a certain expansion. The change starts with the rod having a certain length at a certain time, and ends with the rod having another length at a later time. It is a change *from* a shorter length *to* a longer length, and the change can metaphorically be described as a movement along the physical dimension of length. The length scale (just like Newtonian space) contains in itself no direction; it is a mere scalar order. The change spoken of takes place in time, and if time has a direction then the length change will inherit this

[15] According to my earlier remarks, this potentiality might be regarded either as a for physical theory necessary instrumental fiction or as a real *de re* potentiality.

direction. If there is no temporal direction, there will only be different length magnitudes at different temporal points; there will not be a change of the rod that is *directed* from smaller to longer.

Both movements and length changes are processes in which a physical dimension that has order but lacks internal direction is connected with another ordered physical dimension – time – that is assumed to have an internal direction. It seems to be a general truth that changes require two physical dimensions: (i) the order of what is changing, and (ii) time as an order with direction. Therefore, the directedness of a change is always a *derived* directedness.

The SI system distinguishes between basic and derived physical scalar determinables, but regards the distinction as conventional and pragmatically based; and rightly so since it does not take any notion of ontological dependency into account. I will now from such a determinable-to-determinable point of view discuss vectors. Let me start by dividing the fundamental vector quantities of classical physics into two groups: (a) velocity, momentum, acceleration, and force; (b) electric current, electric field strength, and magnetic field strength.

Looking at the scalar relatives of the first group in the SI system, one finds that momentum is defined as mass times *velocity*, acceleration as rate of change of *velocity*, force as mass times rate of change of *velocity*. Therefore, from a traditional SI perspective, velocity can be regarded as the most basic quantity in the first group. Two seeming counter examples are (i) forces of electrostatic fields and (ii) equilibrium situations where two counteracting forces cancel each other; there is in neither the first nor the second kind of cases any actual velocity. However, forces of electrostatic fields are counterfactually defined by means of the acceleration that they *would* impose on a particle of unit charge, and equilibrium situations can be counterfactually understood by means of the acceleration that arises when only one of the forces is present.

I have argued that velocity is always velocity of a movement, that velocity inherits its directedness from the directedness of the movement in question, and that this latter directedness is founded on the direction of time. Therefore, in a 'vector version' of the SI system that is concerned only with classical mechanics, duration in directed time ought to be – both from the SI systems own pragmatic calculation point of view and an ontological dependence point of view – the basic vector on which the directedness of velocity, momentum, acceleration, and force is to be founded.

What then about the second (electromagnetic) group? The SI definitions of the derived scalar quantities 'electric field strength' and 'magnetic field strength' contain as one part the dimension of 'force', and they are therefore dependent on velocity and directed time, too. But the scalar quantity ‘electric current’ is a base quantity.[16] Let me now use the determinable-to-determinable calculation conventionality of the SI system that I mentioned in Section 1. Electric charge (which has no vector counterpart) is in the SI a derived scalar quantity: ‘(electric charge) $=_{def}$ (electric current) x (time)’. But, due to the proclaimed calculation conventionality, one can equally well make electric charge basic and turn electric current into a derived quantity: ‘(electric current) $=_{def}$ (electric charge) x $(\text{time})^{-1}$’. From an ontological dependence point of view, it would even be more correct to have electric charge as the basic quantity. There can be no electric currents where there are no moving electric charges, but there can be electric charges without electric currents, which means that electric current is one-sidedly existentially dependent on electric charge. That is, even ‘electric current’ can, as a vector quantity, be regarded as having its directedness founded on the direction of time, and be in conformity with one central view of this paper: *if time is not directed there are no real physical vector quantities, only physical scalar quantities.*

Now, what to say about four-dimensionalism? According to this metaphysical position, time does not flow at all, and can then of course have no direction either. It implies, among other things, that theories such as evolutionary cosmology and biology should not be taken literally. On the assumption that my analysis of the classical physical vectors is correct, it also implies that classical vector expressions have a non-mathematical content only in their scalar quantity part. Therefore I claim: what the true metaphysics of physical vectors look like is intimately connected with what the true metaphysics of time looks like.

3. Summary of the analysis of vectors in classical physics

Where mathematics deals with *numbers* and *vectors* mathematical physics deals with *scalar quantities* and *vector quantities*, respectively; and these two kinds of entities are ontologically different. A quantification of a scalar physical dimension contains both the discovery of non-conventional ontological order relations among the corresponding determinates, and a conventional coordinative definition that connects the number one to a specific determinate of the physical dimension. In a similar way, the directedness of vector quantities requires special coordinative definitions that connect mathematical unit vectors to directions in physical

[16] It is also, just like ‘force’, part of the definitions of ‘electric field strength’ and ‘magnetic field strength’.

dimensions. In classical physics, such directedness is dependent on the assumed directedness of time. Of course, no pure mathematical entities are in need of any coordinative definitions.*

References

Dictionary of Mathematics, 2003, London: Penguin.

Hand, D. J. 2004, *Measurement Theory and Practice*, London: Arnold.

Johansson, I. 2000, 'Determinables as Universals', *The Monist* **83**, pp. 101-121.

Johansson, I. 2001, 'Presuppositions for Realist Interpretations of Vectors and Vector Addition', in: U. Meixner, ed., *Metaphysics in the Post-Metaphysical Age.Proceedings of the 22nd International Wittgenstein-Symposium*, Vienna: öbv εt hpt Verlagsgesellschaft, pp. 200-206.

Johansson, I. 2004, *Ontological Investigations* (2 ed.), Frankfurt: Ontos Verlag.

Johansson, I. 2006, 'Review of E. J. Lowe, The Four-Category Ontology', *dialectica* **60**, pp. 513-518.

Johansson, I. 2008, 'Functions and Shapes in the Light of the International System of Units', *Metaphysica* **9**, pp. 93-117.

Johnson, W. E. 1964a, *Logic. Part I*, New York: Dover Publications.

Johnson, W. E. 1964b, *Logic. Part II*, New York: Dover Publications.

Johnson, W. E. 1964c, *Logic. Part III*, New York: Dover Publications.

Kaplan, A. 1964, *The Conduct of Inquiry*, San Francisco: Chandler Publishing Company.

Lowe, E. J. 2005, 'Ontological Dependence', in: *Stanford Encyclopedia of Philosophy*, URL=< http://plato.stanford.edu/entries/dependence-ontological/>.

Pettofrezzo, A. J. 1994, *Vectors and Their Applications*, New York: Dover Publications.

Reichenbach, H. 1958, *The Philosophy of Space and Time*, New York: Dover Publications.

Sanford D. H. 2006, 'Determinates vs. Determinables', in: *Stanford Encyclopedia of Philosophy*, URL=<http://plato.stanford.edu/entries/determinate-determinables/>.

Smith, B. 1994, *Austrian Philosophy*, Peru, Illinois: Open Court Publishing Company.

The International System of Units (SI), 8th edition, 2006, Bureau International des Poids et Mesures URL=<http://www.bipm.org/en/si/si_brochure/>.

* I would like to thank Björn Haglund and two anonymous referees for comments that enabled me to state some of my views more clearly.