Shape is a Non-Quantifiable Physical Dimension

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Abstract. In the natural-scientific community it is often taken for granted that, sooner or later, all basic physical property dimensions can be quantified and turned into a kind-of-quantity; meaning that all their possible determinate properties can be put in a one-to-one correspondence with the real numbers. By using some transfinite mathematics, the paper shows this tacit assumption to be wrong. Shape is a very basic property dimension; but, since it can be proved that there are more possible kinds of determinate shapes than real numbers, shape cannot be quantified. There will never be a shape scale the way we have length and temperature scales. This is the most important conclusion, but more is implied by the proof. Since every n-dimensional manifold has the same cardinality as the real number line, all shapes cannot even be represented in a three-dimensional manifold the way perceivable colors are represented in so-called color solids.

1 Shape as a Physical Dimension

Let me start almost from scratch. The word 'shape' (possible synonyms: 'form' and 'figure') is in everyday life mostly used to refer to the two- and three-dimensional finite outlines of surfaces and things. Such shapes are *closed* shapes, but in science there is often talk also of *open* shapes, e.g., parabolas and hyperbolas. In what follows, when nothing to the contrary is said, the extension of the term 'shape' will include both closed and open shapes with finite spatial extension. Moreover, not only geometrical such shapes, but any arbitrary such shape whatsoever. If a determinate shape (at a certain region in space) by means of translation, rotation, and uniform scaling can be made congruent with another determinate shape at another region, then these two shapes are two instances of the same (kind of) determinate shape. In other words, shapes are invariant to location, rotation, and size.

Shape so delineated is a property dimension on a par with basic physical property dimensions such as length, mass, and temperature. Some determinate shapes also have second-order properties such as being symmetric, being regular, and being polyhedral; and some have mathematical properties such as specific eccentricities.

Ever since the birth of modern science, shapes have been of considerable significance in several disciplines. In astronomy, the shapes of the orbits of planets and

asteroids have always been of central interest; and in classifications of crystals, plants and animals, shape has always been one among the features used. Famously, the fact that the DNA molecule has the shape of a double helix is crucial when it comes to understanding how it functions, but the same is true of most protein macromolecules. Structural biology has even become a special branch of molecular biology. In medicine, shape can play quite a role when it comes to evaluating what is seen by means of X-ray pictures, computer tomography, magnetic resonance imaging, and functional neuroimaging. In short, the natural sciences and the life sciences have always been referring to shapes without quantifying this feature in the way the basic property dimensions of mathematical physics have been quantified, i.e., that each and every possible determinate of the property dimension in question is (via a conventionally chosen standard unit) put in a one-to-one correspondence with a real number.

That shape, despite not being quantified, is commonly regarded as a physical property dimension with many different determinate shapes is obvious. In ordinary language, determinate shapes are represented by means of words such as 'round', 'elliptical', 'triangular', and 'star-shaped'; but we can also represent determinate shapes by means of pictures. As a matter of fact, shape is today in the natural sciences an important but nonetheless non-quantified physical dimension; I will show that this is not a contingent fact. (The paper develops thoughts earlier put forward in [1].)

The quantifications of length, mass, temperature, etc. are quantifications of all the possible determinate properties of the physical dimension in question; for instance, there are no possible determinate mass properties outside of the quantity dimension mass. Therefore, with respect to shape, the quantification problem to be dealt with here is whether or not *all possible shapes* can be quantified.

Surely, at least one subset of shapes, the ellipses, can be linearly ordered and mapped onto a part of the number line. This is done by means of their eccentricity (e). If the length of the semimajor axis of the ellipse is called a and the length of the semiminor axis b, then each and every ellipse can be given an eccentricity value larger than zero but lesser than one according to the formula: $e = \sqrt{(1 - b^2/a^2)}$. Such orderings can be very useful and important for specific purposes in science, but this is beside the quantification problem now at issue. What has been done with respect to ellipses cannot, I claim, be done for all possible shapes. Whereas it makes good sense to say 'this ellipse has an eccentricity of 0.73', it is impossible to give every shape a number related to a standard unit (call it 'morphe'), and make sense of sentences such as 'this shape has a morphe of 2.31'. In my non-quantifiability proof, I will in detail only discuss finite 2D-shapes; if these cannot be quantified neither can the more complex finite 3D-shapes.

2   Shapes and their Segments

Every 2D-shape has some two-dimensional extension in a real or an abstract space, and the shape outline can always be divided into a number of segments. Furthermore, since the shape/line in question constitutes a continuum, it must contain at least as many infinitesimally small segments as there are real numbers.

The next thing to be noted is that there are different kinds of shape segments. Using an idea from Hoffman and Richards [2], I will do some of the reasoning on the assumption that there are five basic different kinds of such segments, but I will later comment on this assumption. The five kinds of segments are:

1. curving-in (⊃)
2. curving-out (⊂)
3. angle-inward (>)
4. straight (−)
5. angle-outward (<).

On the assumption made, each infinitesimal segment of a two-dimensional shape can be ascribed one and only one of these five different kinds of open shapes. Even though the curving-in by rotation can be turned into the curving-out (and the angle-inward into the angle-outward), and they are in this sense identical shapes, they are nonetheless *as shape segments* different. If a curving-in is via one of its end points joined to the end point of another curving-in of the same kind, one obtains another shape than if it is joined to a corresponding curving-out.

I now claim: *there are both as many different determinate curving-ins and curving-outs as there are real numbers*. Furthermore, there are also, taken together, as many segments of kinds 3 to 5 as there are real numbers. This can be shown as follows.

First, let us look at the curving-ins at the vertex of hyperbolas. Like ellipses, hyperbolas can be linearly ordered by means of an eccentricity measure, and each different eccentricity implies a different curving-in at the vertex. According to the usual eccentricity formula for hyperbolas, eccentricity can vary from >1 to infinity, which means that there are at least as many curving-ins as there are real numbers between 1 and infinity. Second, parabolas have another curvature at the vertex than the hyperbolas; let us associate all these parabola curvatures with the number 1, since they have the same eccentricity, 1. Third, let us look at the curvatures at the end points of the major axis of the ellipses. They differ from that of the hyperbolas and parabolas, and they also differ from each other depending on the eccentricity of the ellipse. This eccentricity varies from zero (the circle) to <1. Surely, adding the curving-ins of hyperbolas, parabolas, and ellipses together, we will obtain at least as many different determinate curving-ins as there are real numbers. And there are of course as many determinate curving-outs as there are curving-ins.

The angle-inwards and the angle-outwards can together with the straight line be linearly ordered according to *degree of angularity*, and be given values between 0 and 360. One merely ascribes angle-inward degrees that are larger than zero but smaller than 180º, gives the straight line the value 180º, and angle-outward degrees larger than 180º but smaller than 360º. Since there are as many values between 0 and 360 as there are real numbers, there as many segments of kinds 3 to 5 as there are real numbers, too.

There are, no doubt, at least as many different kinds of possible shape segments as there are real numbers.

The Hoffman-Richards view does not take fractal geometry into account, but this does not affect my reasoning. Let me explain. Every given distinctly perceivable shape outline can be divided into Hoffman-Richards segments. However, we may regard our perception of the shape outline to be too coarse-grained to show the actual shape. If we zoom in on the line to a more fine-grained level, we may discover that it in fact has

another shape than we at first thought. Now, since the identity of a shape is invariant to uniform scaling, the following is a possible scenario. The first-level shape is the arc ⌒, and the second-level shape consists of seven connected exactly similar but much smaller arcs (⌒⌒⌒⌒) bent in an arc. Iterating the zooming in procedure to an even more fine-grained level shows that each of the small second-level arcs consists, in turn, of seven exactly similar but even smaller arcs. And so on, into infinitesimally small arcs.

This infinite sequence gives rise to an interesting question: should the whole bent seven-elements arc be regarded as a basic shape segment in itself, or should it be regarded as consisting of seven instances of an Hoffman-Richards segment? This paper, however, requires no answer to the question. The positing of irreducibly fractal shape segments would not alter the fact that there are as many different kinds of possible Hoffman-Richards segments as there are real numbers; and that is all I need for my reasoning in the next section.

3   There are More Shapes Than Real Numbers

Let me repeat: there are at least as many different possible kinds of shape segments as there are real numbers. The real task, however, is *to prove that there are more possible shapes than real numbers*. In order to do this, some transfinite mathematics is needed. Two things should be remembered:

- Raising 2 to the power of a cardinal, be it finite (n) or infinite ($\aleph$), always gives a larger cardinal: $2^n > n$ and $2^\aleph > \aleph$.
- The set of real numbers have a cardinality that is larger than that of the set of natural numbers; if the latter is called aleph-null ($\aleph_0$), the former is $2^{\aleph_0}$.

If the much discussed continuum hypothesis is true, then there is no cardinal number between $\aleph_0$ and $2^{\aleph_0}$, and if it is false then there is at least one. In the former case, the cardinality of the real numbers ought to be called aleph-one ($\aleph_1$), but if the hypothesis is false, it ought to be called at least aleph-two ($\aleph_2$). At no point in my reasoning, however, does the truth or falsity of the continuum hypothesis play any role; but instead of avoiding an aleph-number altogether, I have chosen to call the cardinality of the set of real numbers $\aleph_1$.

Let us now take a look at what 2D-shapes (lines) may look like when analyzed into a number of end-to-end connected segments that might be shape segments of different kinds. Shapes that consist of two segments chosen among two different kinds of shape segments (a, b) can, following ordinary combinatorial reasoning, be combined in four different ways. But there are nonetheless not four, but three different shapes. This is due to the fact that a 2D-shape is invariant to rotation (rotation in a third dimension is allowed). If we represent the four possible combinations as <a, a>, <b, b>, <a, b>, and <b, a>, then the last two combinations can by rotation be made identical. Similarly, if we look at two particular segments chosen among three different kinds of segments (a, b, c), then we obtain nine combinations, namely <a, a>, <b, b>, <c, c>, <a, b>, <b, a>, <a, c>, <c, a>, <b, c>, <c, b>, but the last six ordered pairs represent only three different shapes.

Only 2-segment combinations that have the same kinds of elements can by rotation be made identical; which means that in this kind of combinatorial enterprise there will always be more shapes than the total number of possible combinations divided by two. In the case of 2 segments chosen among 2 kinds-of-segment, there are more than $2^2/2$ shapes (namely 3); in the case of 2 segments chosen among 3 kinds-of-segment there are more than $3^2/2$ shapes (namely 6); 4 kinds-of-segment give more than $4^2/2$ shapes; and so on. If we look at shapes consisting of three segments, we obtain, for two, three, and four different kinds-of-segment more than $2^3/2$ (namely 6), $3^3/2$ (namely 18), and $4^3/2$ shapes, respectively. In the case of three segments out of three different kinds-of-segment, the segment <a, b, c> can by rotation be turned into <c, b, a>, but not into <c, a, b>; no rotation can turn the middle segment into an end segment. In general, out of a finite number of segments, m, chosen among n different kinds-of-segment, more than $n^m/2$ different possible shapes can be produced. Letting n and m move into infinity does not, as far as I know, change anything in the remarks made.

From what has been said the following ensues. If we fix the number of *segments* to 2, and then let the number of kinds-of-segment progress from 2, 3, and 4 to an arbitrary finite number n, and further into the infinite cardinal numbers $\aleph_0$ and $\aleph_1$, it holds true that we obtain, in turn, at least the following number of shapes: $2^2/2$, $3^2/2$, $4^2/2$, $n^2/2$, $\aleph_0^2/2$, and $\aleph_1^2/2$ (this progression makes up the uppermost line in Table 1). However, since $\aleph_0^2/2 = \aleph_0^2 = \aleph_0$ and $\aleph_1^2/2 = \aleph_1^2 = \aleph_1$, we do not here find more shapes than real numbers. Let us try another course (resulting in the first column in Table 1).

Now, we first fix the number of *kinds-of-segment* to 2, and then let the number of segments progress from 2, 3, and 4 to an arbitrary finite number m, and further into the infinite cardinal numbers $\aleph_0$ and $\aleph_1$. With respect to the finite numbers there are at least, in turn, the following numbers of shapes: $2^2/2$, $2^3/2$, $2^4/2$, and $2^m/2$. Aleph-null number of segments gives us $2^{\aleph_0}/2$ number of shapes. Since $2^{\aleph_0}/2 = 2^{\aleph_0} > \aleph_0$, we have found a set of shapes with a cardinal number larger than that of the natural numbers, but we have not yet found a set of shapes whose cardinal number is provably larger than the set of real numbers ($\aleph_1$). However, aleph-one number of segments supplies us with such a set. Even if there are only 2 kinds-of-segment possible, but there are $\aleph_1$ number of segments, there are more than $\aleph_1$ number of shapes, since $2^{\aleph_1}/2 = 2^{\aleph_1} > \aleph_1$.

The reasoning in the last two paragraphs can be repeated in such a way that we obtain the results displayed in Table 1. In all the squares of the lowest line, but nowhere else, there are more possible shapes than real numbers.

	2 kinds:	3 kinds:	4 kinds:	n kinds:	$\aleph_0$ kinds:	$\aleph_1$ kinds:
2 seg.:	$2^2/2$	$3^2/2$	$4^2/2$	$n^2/2$	$\aleph_0^2/2$	$\aleph_1^2/2 = \aleph_1^2 = \aleph_1$
3 seg.:	$2^3/2$	$3^3/2$	$4^3/2$	$n^3/2$	$\aleph_0^3/2$	$\aleph_1^3/2 = \aleph_1^3 = \aleph_1$
4 seg.:	$2^4/2$	$3^4/2$	$4^4/2$	$n^4/2$	$\aleph_0^4/2$	$\aleph_1^4/2 = \aleph_1^4 = \aleph_1$
m seg:	$2^m/2$	$3^m/2$	$4^m/2$	$n^m/2$	$\aleph_0^m/2$	$\aleph_1^m/2 = \aleph_1^m = \aleph_1$
$\aleph_0$ seg:	$2^{\aleph_0}/2 > \aleph_0$	$3^{\aleph_0}/2 > \aleph_0$	$4^{\aleph_0}/2 > \aleph_0$	$n^{\aleph_0}/2 > \aleph_0$	$\aleph_0^{\aleph_0}/2 > \aleph_0$	$\aleph_1^{\aleph_0}/2 = \aleph_1^{\aleph_0} = \aleph_1$
$\aleph_1$ seg:	$2^{\aleph_1}/2 > \aleph_1$	$3^{\aleph_1}/2 > \aleph_1$	$4^{\aleph_1}/2 > \aleph_1$	$n^{\aleph_1}/2 > \aleph_1$	$\aleph_0^{\aleph_1}/2 > \aleph_1$	$\aleph_1^{\aleph_1}/2 = \aleph_1^{\aleph_1} > \aleph_1$

Table 1. The lines tell how many shape segments a certain shape has, and the columns how many kinds-of-segment the same shape has. The number in a specific square tells how many different kinds of shapes the given combination (of number of segments and number of kinds-of-segment) can surely give rise to.

I have for a long time suspected that the fact that shape has not been quantified is not due to contingency or lack of intellectual capability among people analyzing shapes. Therefore, I am not astonished that it is possible to put forward a proof that this cannot be done. I am, however, astonished that that there are more shapes than real numbers already in the lower left corner in Table 1. Originally, I thought this would be the case only in the lower right corner. It should, however, be remembered that transfinite mathematics often gives astonishing results. Table 1 makes it clear that the existence of two different kinds-of-segment is enough for the proof; and such a minimal assumption seems incontestable.

4 Shape Cannot be Quantified

In a quantification of a scalar property dimension, the determinate properties of a physical property dimension are given a one-to-one mapping onto the real numbers. Now, if a domain S has a larger cardinality than another domain R, then there can be no one-to-one mapping between S and R. Therefore, if there are a larger number of 2D-shapes (S) than there are real numbers (R), then 2D-shapes cannot possibly be quantified. That is, the proof that there are more 2D-shapes than real numbers entails a proof that all possible 2D-shapes cannot be quantified. And since there are at least as many 3D-shapes as there are 2D-shapes, neither can all possible 3D-shapes be quantified.

In the proof that there are more shapes than real numbers no distinction is made between closed and open shapes. The combinatorial reasoning used is indifferent to whether segment combinations give rise to closed or to open shapes. Now, it might be argued, that just as length and area are regarded as different physical dimensions, open and closed shapes should similarly, for some topological reasons, be regarded as two different physical dimensions. And, if this is the case, it might be said that I have not really proved that shape cannot be quantified; only proved that the pseudo-dimension consisting of both closed and open shapes cannot be quantified.

I find it hard, however, to see that open and closed shapes can be regarded as two different property dimensions. This would mean that turning an open curved line into a circle would change its property dimension. But more can be said.

Since only two different kinds-of-segment are needed for the proof, it goes through for the subset of open shapes if there are two different kinds-of-segment all whose combinations give rise to open shapes. And there are. Every combination of one curving-in segment with one curving-out segment results in an open shape. That is, not only is the set of all shapes not quantifiable; the same is true for the subset of open shapes, too. I am not, however, able to prove that there are more closed shapes than real numbers.

5 Other Implications of the Proof

I have so far used 'quantification' in its (outside of logic) central sense, i.e., to quantify a kind of property is to make it possible to assign numbers to all its determinates in such

a way that the linear order of the real numbers correspond to a linear order of the determinates. Sometimes, however, the term takes on such a wide meaning that everything that can be mapped onto a part of an n-dimensional vector space is said to have been quantified. For instance, all humanly perceivable colors have been given a number of different quite elegant and informative mappings by means of three-dimensional so-called color solids; the dimensions being hue, saturation/chroma, and intensity/value. Is it perhaps possible to map all possible shapes into a corresponding "shape solid"?

The answer to the question is: no, it is not. The reason is that any specific n-dimensional manifold has the same number of points as there are points in the real number line. This general truth implies that since there are more shapes than real numbers, all shapes cannot even be mapped into an n-dimensional solid; whatever value n is given.

A further and quite peculiar thing follows from the general truth just mentioned. I have been discussing possible shapes. The proof states that there are more *possible* determinate shapes than there are real numbers. But, as just pointed out, this means that there are more possible shapes than there are points in *any* n-dimensional manifold, not only when the manifold is a line and n equals 1. The conclusion to draw is this: independently of whether we regard our world as being three-dimensional, four-dimensional, eleven-dimensional (as one kind of string theory claims) or something else, all *possible* shapes can never exist actually.

6   Concluding Comments

The non-quantifiability proof put forward must not be misunderstood. Without the addition of further premises, it does by no means imply that subsets of shapes cannot be quantified. And, of course, wherever and whenever a partial quantification of shape fulfills a scientific or practical purpose, it should be made.

Neither does the proof imply that the shape dimension exists in splendid isolation from natural laws stated by means of variables that relate quantified physical property dimensions to each other. What the special theory of relativity says about length contraction has implications for what is true about shapes. Just one example: necessarily, what is a circle in one specific inertial frame is a non-circle in all other inertial frames.

Note, again, that it doesn't matter to the proof put forward whether or not the much discussed continuum hypothesis is true or false. It does, however, rely on two assumptions that are not obviously true:

- Traditional combinatorial reasoning can be used in relation to combinations of infinitesimally small shape segments.
- Finite 2D-shapes can be divided into $2^{\aleph_0}$ shape segments.

Neither of these assumptions seems ever to have been seriously scrutinized; and I don't know how to prove them. Moreover, perhaps it is the case that they cannot be strictly proved; and neither strictly disproved. Such a proof or counter-proof may require that shape has already been quantified, which I have claimed is impossible. This

does *not* mean, however, that I cannot be proved to be wrong. As the proof of the pudding is only in the eating, and not in a thinking that tries to find out how the pudding tastes; the proof of the possibility of a quantification of shape may consist only in an actual quantification, and not in an attempt to disprove the somewhat problematic assumptions mentioned.

Leaving this remark aside, let us nonetheless assume that one day one of these assumptions, or perhaps another one not noted by me, is shown to be false. What would such a discovery imply?

Of course, it would show that I have managed to prove neither that the number of shapes is larger than aleph-one nor that shape is a non-quantifiable dimension. It would *not*, however, show that shape really is quantifiable. In order to prove this, at least two further tasks have to be carried out. First, a proof to the effect that there cannot possibly be more shapes than real numbers has to be construed. Second, it has to be shown that it is possible to order all possible shape determinates linearly, which seems to run counter to the intuition that determinate shapes have no magnitude, degree, or intensity.

In the natural-scientific community it is often simply taken for granted that, sooner or later, all basic physical property dimensions, shape included, can be quantified. I think the default view today ought to be the opposite: shape is a non-quantifiable physical property dimension.

Acknowledgements

References

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